ITERATION-DISCRETIZATION METHODS FOR SOME VARIATIONAL INEQUALITY

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Abstract

Iterative methods for solving variational inequalities in infinite dimensional Hilbert spaces as a rule require some discretization. This leads to variational inequalities over families of spaces. In the present paper this problem is addressed by an iterative method with only a finite number of steps at each discretization level. First, abstract methods are studied and later an optimal control problem with elliptic state equations and some bound on the controls is considered. The discretization technique rests upon a nested family of piecewise linear C^0-elements conforming finite element discretizations.

Keywords: Optimal control; iteration-discretization; projection algorithm; elliptic state equation; finite element discretization.

Introduction

The solution of variational inequalities in function spaces often requires discretizations as well as iteration methods for solving the obtained finite dimensional problems. Similarly, the practical application of iteration methods in function spaces as a rule needs some finite dimensional approximation of the iteration procedure. Both processes, i.e. discretization and iteration, are not finite. The aim of our paper is to provide a sketch of the ideas developed in [3] for a new iteration-discretization method for variational inequality problems over the fixed point set of a quasi-nonexpansive operator which bases on appropriate extensions to families of problems and mappings. We give partially an overview over the problem setting and illustrate underlying analytical results. The related proofs can be found in [3].

1 Variational Inequalities over Sets of Fixed Points

Let $\mathscr{H}$ denote a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the related norm $\|\cdot\|$ and $T:\mathscr{H}\to\mathscr{H}$ be a quasi-nonexpansive operator, i.e. an operator with $\operatorname{Fix} T\neq\emptyset$ and

$$\|Tu-z\|\leq\|u-z\| \qquad \text{for all } u\in\mathscr{H}, \quad \text{for all } z\in\operatorname{Fix} T,$$

where

$$\operatorname{Fix} T := \{v\in\mathscr{H} : Tv=v\}.$$

Further, let a mapping $\mathscr{F}: \mathscr{H} \to \mathscr{H}$ be given which is κ-Lipschitz continuous and η-strongly monotone over $\mathscr{H}$, i.e. with some $\kappa \geq \eta > 0$ holds

$$\|\mathscr{F}(u) - \mathscr{F}(v)\| \leq \kappa \|u - v\| \qquad \text{for all } u, v \in \mathscr{H} \tag{1}$$

and

$$\langle \mathscr{F}(u) - \mathscr{F}(v), u - v \rangle \geq \eta \|u - v\|^2 \qquad \text{for all } u, v \in \mathscr{H}. \tag{2}$$

We consider the problem, called $VIP(\mathscr{F}, \mathrm{Fix}\, T)$,

Find $\bar{u} \in \mathrm{Fix}\, T$ *with*

$$\langle \mathscr{F}(\bar{u}), u - \bar{u} \rangle \geq 0 \text{ for all } u \in \mathrm{Fix}\, T. \tag{3}$$

Since $\mathrm{Fix}\, T$ is non-empty, closed and convex there exists a unique solution $\bar{u}$ of the problem $VIP(\mathscr{F}, \mathrm{Fix}\, T)$. Problems of this type are considered in e.g. [1], [5], [8] and the book [2] provides a comprehensive discussion of them.

Having infinite-dimensional Hilbert spaces $\mathscr{H}$ in mind, the numerical treatment of $VIP(\mathscr{F}, \mathrm{Fix}\, T)$ requires an appropriate discretization. Let $\mathscr{H}_k \subset \mathscr{H}$ denote a family of nested closed subspaces and $T_k : \mathscr{H} \to \mathscr{H}_k$ a family of quasi-nonexpansive operators that satisfy $\mathrm{Fix}\, T_k \subset \mathrm{Fix}\, T$ and $\bigcap_{k=0}^{\infty} \mathrm{Fix}\, T_k \neq \emptyset$. Further, let $\{\mathscr{F}_k\}_{k=0}^{\infty}$ with $\mathscr{F}_k : \mathscr{H} \to \mathscr{H}_k$ be a sequence of operators which are κ-Lipschitz continuous and η-strongly monotone over $\mathscr{H}_k$. Now, instead of $VIP(\mathscr{F}, \mathrm{Fix}\, T)$ we will consider the following sequence of problems $VIP(\mathscr{F}_k, \mathrm{Fix}\, T_k)$

Find $\bar{u}^k \in \mathrm{Fix}\, T_k$ *with*

$$\langle \mathscr{F}_k(\bar{u}^k), u - \bar{u}^k \rangle \geq 0 \text{ for all } u \in \mathrm{Fix}\, T_k. \tag{4}$$

The corresponding conditions for the approximation will be presented in the further part of the paper (see Theorem 1). As for $VIP(\mathscr{F}, \mathrm{Fix}\, T)$ the made assumption guarantees that each of the problems $VIP(\mathscr{F}_k, \mathrm{Fix}\, T_k)$ possess a unique solution $\bar{u}^k$. However, we will not solve each problem $VIP(\mathscr{F}_k, \mathrm{Fix}\, T_k)$, but propose and analyze an iteration-discretization procedure that approximate the solution of $VIP(\mathscr{F}, \mathrm{Fix}\, T)$ by simultaneously performing iteration steps and refining the discretization.

2 Iteration Methods Based upon Families of Operators

With $\mu \in (0, \frac{2\eta}{\kappa^2})$ and $\{\lambda_k\}_{k=0}^{\infty} \subset (0, \mu]$ we consider the following iteration for solving $VIP(\mathscr{F}, \mathrm{Fix}\, T)$:

$$u^{k+1} = (I - \lambda_k \mathscr{F}_k)\, T_k u^k, \tag{5}$$

where I denotes the identity operator and $u^0 \in \mathscr{H}$ is arbitrarily chosen. An alternative recursion is defined by

$$u^{k+1} = T_k (I - \lambda_k \mathscr{F}_k)\, u^k, \tag{6}$$

which we study later under the additional assumption that the operators T_k are projections on closed convex sets. Both iterations can be seen equivalent after transformations. Despite the fact that the sequences generated either by (5) or by (6) do not coincide we use the same notation since in the sequel we clearly distinguish which of the iterations is applied. First, we analyze the iteration (5). Denote

$$S_k = (I - \lambda_k \mathscr{F}_k) T_k. \tag{7}$$

Iteration (5) can be shortly written in the form

$$u^{k+1} = S_k u^k.$$

Let $\mu \in \left(0, \frac{2\eta}{\kappa^2}\right)$, define

$$\tau := 1 - \sqrt{1 + \mu^2\kappa^2 - 2\mu\eta}. \tag{8}$$

Since $\eta \leq \kappa$, consequently, τ is well defined and we have $\tau \in (0,1]$.

Lemma 1 *Let $\mu \in \left(0, \frac{2\eta}{\kappa^2}\right)$. Then the operators $G_k := I - \mu\mathscr{F}_k$ satisfy*

$$\|G_k x - G_k y\| \leq (1-\tau)\|x-y\| \qquad \textit{for all} \quad x,y \in \mathscr{H}_k.$$

From the lemma above with (7) we obtain

Corollary 1 *Let $\mu \in \left(0, \frac{2\eta}{\kappa^2}\right)$ and $\lambda_k \in [0,\mu]$. Then*

$$\|S_k u - S_k v\| \leq \left(1 - \frac{\lambda_k \tau}{\mu}\right)\|T_k u - T_k v\| \qquad \textit{for all} \quad u,v \in \mathscr{H}_k.$$

For the further analysis we make the assumption:

$$\text{There exist } z \in \bigcap_{k=0}^{\infty} \operatorname{Fix} T_k \text{ and some } c \in \mathbb{R} \text{ such that } \|\mathscr{F}_k z\| \leq c \tag{9}$$

for all $k \geq 0$. The κ-Lipschitz continuity of $\mathscr{F}_k$ for any $r > 0$ yields the boundedness of $\|\mathscr{F}_k u\|$ for any $u \in B(z,r) := \{v \in \mathscr{H}_k : \|v - z\| < r\}$.

Lemma 2 *Let $u^0 \in \mathscr{H}$ be arbitrary and $\{u^k\}_{k=0}^{\infty}$ be generated by (5). Then the sequences $\{u^k\}_{k=0}^{\infty}$, $\{\mathscr{F}_k T_k u^k\}_{k=0}^{\infty}$ and $\{T_k u^k - P_{\operatorname{Fix} T}\, u^k\}_{k=0}^{\infty}$ are bounded, where $P_{\operatorname{Fix} T}$ denotes the metric projection onto* $\operatorname{Fix} T$.

Define

$$\alpha_k := \frac{\lambda_k \tau}{\mu} \tag{10}$$

and

$$\beta_k := \alpha_k \frac{\mu^2}{\tau^2}\left(\|\mathscr{F}_k \bar{u}^k\|^2 + 2\langle \mathscr{F}_k T_k u^k - \mathscr{F}_k \bar{u}^k, \mathscr{F}_k \bar{u}^k\rangle\right) \tag{11}$$

It is clear that $\alpha_k \in (0,1]$.

Lemma 3 *Let u^{k+1} be given by (5) and $\bar{u}^k \in \operatorname{Fix} T_k$ be the unique solution of VIP$(\mathscr{F}_k, \operatorname{Fix} T_k)$. Then*

$$\|u^{k+1} - \bar{u}^k\|^2 \leq (1-\alpha_k)\|u^k - \bar{u}^k\|^2 + \alpha_k \beta_k \tag{12}$$

Before we turn to the convergence theorem we provide the following auxiliary result.

Lemma 4 *Let $\{a_k\}_{k=0}^{\infty} \subset \mathbb{R}_+$ be a sequence satisfying the inequality*

$$a_{k+1} \leq (1-\alpha_k)a_k + \alpha_k \beta_k + \gamma_k, \tag{13}$$

where $\{\alpha_k\}_{k=0}^{\infty} \subset [0,1]$, $\{\beta_k\}_{k=0}^{\infty} \subset \mathbb{R}_+$, $\{\gamma_k\}_{k=0}^{\infty}, \subset \mathbb{R}_+$. *If* $\sum_{k=0}^{\infty} \alpha_k = +\infty$, $\limsup_k \beta_k \leq 0$ *and* $\sum_{k=0}^{\infty} \gamma_k < +\infty$ *then*

$$\lim_{k\to\infty} a_k = 0.$$

Theorem 1 *Let $\bar{u} \in \mathrm{Fix}\, T$ be the unique solution of VIP$(\mathscr{F}, \mathrm{Fix}\, T)$ and let $\bar{u}^k$ be the unique solution of VIP$(\mathscr{F}_k, \mathrm{Fix}\, T_k)$. Suppose that $\{\mathscr{F}_k \bar{u}^k\}_{k=0}^{\infty}$ is bounded and that*

$$\sum_{k=0}^{\infty} \|\bar{u}^k - \bar{u}\| < +\infty. \tag{14}$$

Let $\{\lambda_k\}_{k=0}^{\infty} \subset (0,\mu]$ be a sequence with

$$\lim_{k\to\infty} \lambda_k = 0 \quad \textit{and} \quad \sum_{k=0}^{\infty} \lambda_k = +\infty. \tag{15}$$

Then for any $u^0 \in \mathscr{H}$ the sequence $\{u^k\}_{k=0}^{\infty}$ generated by (5) converges strongly to $\bar{u}$.

3 Iterations by a Sequence of Contraction Mappings

Now, we study the convergence behavior of the alternative iteration process (6), i.e. the case where $\{u^k\}_{k=0}^{\infty}$ is generated by the iteration

$$u^{k+1} = V_k u^k := T_k\,(I - \lambda_k \mathscr{F}_k)\,u^k. \tag{16}$$

For this type of iteration we apply metric-projections as operators T_k. These operators are non-expansive. Unlike in the preceeding section here the parameters λ_k have not to tend to zero. For the operators $V_k = T_k\,(I - \lambda_k \mathscr{F}_k)$ with Lemma 1 for $\lambda_k := \mu$ we obtain

$$\|V_k u - V_k v\| \leq \sigma \|u - v\| \qquad \text{for all} \quad u, v \in \mathscr{H}_k, \tag{17}$$

where $\sigma := 1 - \tau < 1$. Further, the map $V := T(I - \lambda \mathscr{F})$ is assumed to satisfy

$$\|Vu - Vv\| \leq \sigma \|u - v\| \qquad \text{for all} \quad u, v \in \mathscr{H}. \tag{18}$$

As a consequence the operators V_k and V possess unique fixed points $\bar{u}^k$ and $\bar{u}$, respectively, i.e.,

$$\bar{u}^k = V_k \bar{u}^k, \quad k = 0,1,2,\ldots \qquad \text{and} \qquad \bar{u} = V\bar{u}. \tag{19}$$

Theorem 2 *Let the conditions (17), (18) be satisfied and let $\bar{u}$ and $\bar{u}^k$ denote the unique fixed points of the operator V and V_k, respectively. Assume that the approximation property*

$$\sum_{k=0}^{\infty} \|\bar{u}^k - \bar{u}\| < +\infty \tag{20}$$

holds. Then for any $u^0 \in \mathscr{H}_0$ the sequence $\{u^k\}_{k=0}^{\infty}$ generated by (16) converges to the fixed point $\bar{u}$ of V.

Proof With the fixed point property of $\bar{u}^k$ for the operator V_k holds

$$\|u^{k+1} - \bar{u}^k\| = \|V_k u^k - V_k \bar{u}^k\| \le \sigma \|u^k - \bar{u}^k\|.$$

With the triangle inequality this yields

$$\|u^{k+1} - \bar{u}^{k+1}\| \le \|u^{k+1} - \bar{u}^k\| + \|\bar{u}^k - \bar{u}^{k+1}\| \le \sigma \|u^k - \bar{u}^k\| + \|\bar{u}^k - \bar{u}^{k+1}\|.$$

and we obtain

$$\|u^{k+1} - \bar{u}^{k+1}\| \le \sigma \|u^k - \bar{u}^k\| + \|\bar{u}^k - \bar{u}\| + \|\bar{u}^{k+1} - \bar{u}\|, \quad k \ge 0. \tag{21}$$

Let define $a_k := \|u^k - \bar{u}^k\|$, $\alpha_k := 1 - \sigma$, $\beta_k := 0$ and

$$\gamma_k := \|\bar{u}^k - \bar{u}\| + \|\bar{u}^{k+1} - \bar{u}\|.$$

Then (21) can be expressed by (13). Trivially $\sum\limits_{k=0}^{\infty} \alpha_k = +\infty$ and $\limsup_k \beta_k \le 0$. The made assumption (20) yields $\sum\limits_{k=0}^{\infty} \gamma_k < +\infty$. Now, we can apply Lemma 4 and obtain $\lim\limits_{k\to\infty} \|u^k - \bar{u}^k\| = 0$. With (20) this completes the proof. ■

4 An Optimal Control Model Problem

Next, we apply the iteration method introduced above to some variational inequality problem that arises from the optimality criterion of an optimal control problem with bounds upon the controls.

Let $\Omega \subset \mathbb{R}^2$ be some open convex polyhedron and Γ its boundary. Consider the optimal control problem

$$\tilde{J}(y,u) := \frac{1}{2} \int\limits_{\Omega} (y-d)^2 + \frac{\alpha}{2} \int\limits_{\Omega} u^2 \to \min ! \tag{22a}$$

$$\text{s.t.} \ -\Delta y = u \text{ in } \Omega, \quad y + \frac{\partial y}{\partial n} = 0 \ \text{ on } \Gamma, \quad u \in Q := \{u : u \le b \ \text{ a.e. in } \Omega\} \tag{22b}$$

with given $\alpha > 0$, $d \in L_2(\Omega)$ and $b \in \mathbb{R}$. The state equation is understood as weak formulation. Let $Y := H^1(\Omega)$ be the Sobolev space of functions over Ω that possess the first order weak derivatives in $L_2(\Omega)$. Further, we let $U = L_2(\Omega)$. Define $a(\cdot,\cdot) : Y \times Y \to \mathbb{R}$ by

$$a(u,v) := \int\limits_{\Omega} \nabla u \cdot \nabla v + \int\limits_{\Gamma} uv \qquad \text{for all } u, v \in Y.$$

Now, the weak formulation of (22b) is given by

$$y \in Y : \qquad a(y,v) = \langle u, v \rangle \qquad \text{for all } v \in Y. \tag{23}$$

The Lax-Milgram Lemma (cf. [4]) guarantees that for any $u \in U$ the equation (23) possesses a unique solution. This defines a linear operator $S : U \to Y$ by

$$Su \in Y : \qquad a(Su,v) = \langle u, v \rangle \qquad \text{for all } v \in Y \tag{24}$$

and with the ellipticity constant $\gamma > 0$ this satisfies

$$\|Su\|_1 \leq \frac{1}{\gamma}\|u\|_0. \tag{25}$$

Using the operator S we obtain the reduced form of (22)

$$J(u) := \frac{1}{2}\langle Su-d, Su-d\rangle + \frac{\alpha}{2}\langle u,u\rangle \rightarrow \text{ min! } \quad \text{s.t.} \quad u \in Q. \tag{26}$$

This problem has a unique solution $\bar{u} \in Q$ and this solution can be characterized by a variational inequality that satisfies the general assumptions made above for the abstract problem (cf. [4], [7]).

Theorem 3 *The problem (26) possesses a unique solution $\bar{u} \in Q$. There $\bar{u} \in U$ forms the solution of (26) if and only if*

$$\langle J'(\bar{u}), u-\bar{u}\rangle \geq 0 \qquad \textit{for all } u \in Q \tag{27}$$

holds.

The structure of $J(\cdot)$ yields

$$J'(u) = S^*(Su-d) + \alpha u, \tag{28}$$

where S^* denotes the adjoint of S given by

$$a(z,w) = \langle z,v\rangle \quad \forall z \in Y \qquad \text{and} \qquad S^*v := w.$$

Now, we can show that (27) is of the considered abstract type. First, the Hilbert space $\mathscr{H}$ is just the space $U = L_2(\Omega)$. The set $Q \subset U$ is closed and convex. Thus, the metric-projection $P_Q : U \rightarrow Q$ is well defined by

$$P_Q u \in Q: \qquad \|P_Q u - u\|_0 \leq \|v-u\|_0 \qquad \text{for all } v \in Q,$$

and we have

$$u \in Q \quad \Longleftrightarrow \quad u \in \operatorname{Fix} P_Q.$$

From $Q \neq \emptyset$ and from the nonexpansivity of P_Q we obtain that $T := P_Q$ is a quasi-nonexpansive operator. Further, the operator $\mathscr{F} : U \rightarrow U$ is defined by

$$\mathscr{F} u := S^*(Su-d) + \alpha u.$$

Trivially, it is Lipschitz continuous because of $U \hookrightarrow H^2(\Omega)$ this yields

$$\|\mathscr{F} u - \mathscr{F} v\| \leq (\|S^*\|\,\|S\| + \alpha)\,\|u-v\| \qquad \text{for all } u, v \in U$$

and with (25) we obtain

$$\|\mathscr{F} u - \mathscr{F} v\| \leq \left(\frac{1}{\gamma^2} + \alpha\right)\|u-v\| \qquad \text{for all } u, v \in U. \tag{29}$$

Further, we have

$$\langle \mathscr{F} u - \mathscr{F} v, u-v\rangle \geq \alpha\,\|u-v\|^2 \quad \text{for all } u, v \in U.$$

This proves that the problem (27) satisfies all assumptions made for the general problem. As a consequence some parameter $\lambda > 0$ can be found such that the iteration

$$u^{k+1} = T(u^k - \lambda \mathscr{F} u^k), \quad k \geq 0. \tag{30}$$

for any $u^0 \in U$ converges to the optimal solution $\bar{u} \in Q$ of the considered control problem. However, the iteration (30) is in the function space U and requires an appropriate discretization.

5 Families of Conforming Discretizations

Let $U_k \subset U$, $Y_k \subset Y$ we apply a piecewise linear C^0-discretization over nested families $\{\mathscr{T}_k\}$ of uniformly regular triangulations (see [4]). With h_k we denote the maximal diameter of the triangles in $\{\mathscr{T}_k\}$. We denote the related grid points by $\Omega_k := \{x^{k,j}\}_{j=1}^{N_k}$. We have

$$\Omega_k \subset \Omega_{k+1}, \quad k \geq 0. \tag{31}$$

Let $\phi_{k,j} \in C(\bar{\Omega})$ be piecewise linear over $\mathscr{T}_k$ with

$$\phi_{k,i}(x^{k,j}) = \delta_{ij}, \qquad i, j = 1, \ldots, N_k, \quad k \geq 0.$$

This means we assume $\{\phi_{k,j}\}$ to form a Lagrangian basis of U_k as well as of Y_k, where

$$U_k := Y_k := \operatorname{span}\left\{\phi_{k,j}\right\}_{j=1}^{N_k}.$$

With (31) this implies $\quad U_k \subset U_{k+1} \subset \cdots \subset U, \quad Y_k \subset Y_{k+1} \subset \cdots \subset Y.$
For given $u \in U$ the conforming discretization of the state equations have the form:

$$y_k \in Y_k : \qquad a(y_k, v) = \langle u, v \rangle \qquad \text{for all } v \in Y_k. \tag{32}$$

Again, the Lax-Milgram Lemma implies that for any $u \in U$ problem (32) possesses a unique solution. Thus, $S_k u := y_k$ defines linear mappings $S_k : U \to Y_k \subset Y$. Let

$$Q_k := \{u \in U_k \,:\, u \leq b\}.$$

This yields the following discrete problems

$$J_k(u) := \frac{1}{2}\langle S_k u - d, S_k u - d\rangle + \frac{\alpha}{2}\langle u, u\rangle \to \text{ min! } \quad \text{s.t.} \quad u \in Q_k. \tag{33}$$

Problem (33) has a unique solution $\bar{u}_k \in Q_k$. Analogously to the continuous case we define $\mathscr{F}_k : U \to Y_k$ by

$$\mathscr{F}_k u := S_k^*(S_k u - d) + \alpha u \qquad \text{for all } u \in U$$

and $T_k := P_{Q_k}$, the metric-projection onto Q_k. Then

$$u_{j+1} = T_k(u_j - \lambda_k \mathscr{F}_k u_j), \quad j \geq 0. \tag{34}$$

forms an iteration technique on the discretization level k. For sufficiently small $\lambda_k > 0$ the sequence generated by (34) converges to the unique solution $\bar{u}^k$ of the discrete control problem (33). As mentioned in Section 3, there are two different types of iterations. Instead of (34) we may also apply iteration-discretization method

$$u_{k+1} = T_k(u_k - \lambda_k \mathscr{F}_k u_k), \quad k \geq 0. \tag{35}$$

with appropriate parameters $\lambda_k > 0$. This recursion differs from (34) by acting on a family of discrete problems, where on each discretization level only a maximal number of iteration steps is performed. In particular, if the discretization changes in each step then (35) means that only one iteration step is performed per discretization.

Now, we derive that (35) fulfills the assumptions made in Section 3. For the proofs we refer again to [3].

Lemma 5 *Let $U_0 \neq \emptyset$ then $\operatorname{Fix} T_k \neq \emptyset$, $k \geq 0$ holds. Further, we have*

$$\operatorname{Fix} T_k \subset \operatorname{Fix} T_{k+1} \subset \operatorname{Fix} T \qquad k \geq 0.$$

Lemma 6 *The continuous optimal control problem (26) has a unique solution $\bar{u}$ and for any $k \geq 0$ the discrete control problem (33) possesses a unique solution $\bar{u}^k \in Q_k$. Further, there is a constant $c > 0$ with*

$$\|\mathscr{F}_k \bar{u}^k\| \leq c, \quad k \geq 0.$$

Theorem 4 *Let $\bar{u}^k \in Q_k$ denote the solution of the discrete problem (33) and $\bar{u} \in Q$ the solution the original continuous problem (26). Then there exists a constant $c > 0$ such that*

$$\|\bar{u}^k - \bar{u}\| \leq c\, h_k. \tag{36}$$

In principle, the estimate given in Theorem 4 could be refined to

$$\|\bar{u}^k - \bar{u}\| \leq c\, h_k^{3/2}$$

if the technique proposed in [6] is applied to the discretization of the elliptic control problem under consideration. In our application, however, the bound (36) is already sufficient to ensure the convergence for simple refinement strategies. Indeed, if $\mathscr{T}_k$ is generated by subdivisions of all triangles using the midpoints of all edges then we obtain $h_{k+1} = h_k/2$ and consequently $\sum\limits_{k=0}^{\infty} h_k < +\infty$ holds. Thus, the proposed iteration-discretization technique converges.

Conclusion

The present paper provides some theoretical approach to the simultaneous refinement of the discretzation of function spaces and iterations for solving variational inequality problems over the fixed point set of a quasi-nonexpansive operator in Hilbert spaces. This approach forms some basis for implementable algorithms because it avoids nested infinite processes. The increase the efficiency of the discussed methods further improvements, e.g. preconditioning, have to be investigated in the future.

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ITERAČNĚ–DISKRETIZAČNÍ METODY PRO VARIAČNÍ NEROVNOST

Iterační metody pro řešení variačních nerovností v Hilbertových prostorech nekonečné dimenze vyžadují diskretizaci. To vede k řešení posloupnosti variačních nerovností v prostorech konečné dimenze. Tato práce se věnuje iteračním metodám, které vyžadují pouze konečný počet kroků na každé diskretizační úrovni. Nejprve je studována abstraktní úloha a následně konkrétní úloha optimálního řízení s eliptickou stavovou rovnicí a s omezeními na řídící proměnnou. Diskretizace je provedena pomocí posloupnosti do sebe vnořených po částech lineárních, spojitých, konformních konečných prvků.

EIN ITERATIONS-DISKRETISIERUNGS-VERFAHREN FÜR EINE VARIATIONSUNGLEICHUNG

Iterationsverfahren zur Behandlung von Variationsungleichungen in undendlichdimensionalen Hilbert-Räumen erfordern in der Regel eine Diskretisierung. Diese führt auf Variationsungleichungen über einer Familie von Räumen. In der vorliegenden Arbeit wird dieses Problem durch ein Iterationsverfahren mit einer nur endlichen Zahl von Schritten je Diskretisierungsniveau behandelt. Zunächst werden abstrakte Methoden untersucht und später auf ein Problem der optimalen Steuerung mit elliptischen Zustandsgleichungen und Steuerrestriktionen angewandt. Die Diskretisierung erfolgt durch eine sich verfeinernde Familie stückweise linearer, konformer C^0 finiter Elemente.

METODA ITERACYJNO-DYSKRETYZACYJNA DLA NIERÓWNOŚCI WARIACYJNEJ

Metody iteracyjne dla nierówności wariacyjnych w nieskończenie-wymiarowych przestrzeniach Hilberta zazwyczaj wymagają dyskretyzacji. Ta z kolei prowadzi do nierówności wariacyjnych określonych na rodzinie przestrzeni. W niniejszej pracy dla tego problemu stosujemy metodę iteracyjną, w której na każdym poziomie dyskretyzacji przeprowadzanych jest skończenie wiele kroków. Najpierw badamy metody abstrakcyjne, które następnie stosujemy do problemu sterowania optymalnego z eliptycznym równaniem stanu i z ograniczeniami na sterowanie. Przy dyskretyzacji stosujemy rodzinę zagęszczających się elementów skończonych, które są kawałkami liniowe, ciągłe i konforemne.