

ADAPTIVE WAVELET SCHEME FOR CONVECTION–DIFFUSION EQUATIONS

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Abstract

The paper is concerned with theoretical and computational issues of a numerical resolution of the convection-diffusion equation. We use an implicit scheme for the time discretization and an adaptive wavelet-based method for a spatial discretization. We use a well-conditioned cubic spline-wavelet basis and a method for an inexact multiplication of wavelet stiffness matrix with a vector which we have recently proposed in [1, 2]. The theoretical advantages of our scheme as well as numerical examples will be presented.

Keywords: Convection-diffusion equation; spline; wavelet; adaptivity.

Introduction

In this paper we consider a numerical approximation of the convection-diffusion equation

$$\frac{\partial u}{\partial t} = \mu \Delta u - b \operatorname{div} u - cu + f \quad \text{for } x \in \Omega, \quad t \in (0, T), \quad (1)$$

with initial and boundary conditions

$$u(x, 0) = u^0(x) \quad \text{for } x \in \Omega, \quad u(x, t) = 0 \quad \text{for } x \in \partial\Omega.$$

We assume that $\mu > 0$, b and c are constants, $u : \Omega \times (0, T) \rightarrow \mathbb{R}$, $f, u^0 \in L^2(\Omega)$. We consider only the domain $\Omega = (0, 1)^n$. It is well-known that the solution of (1) typically contains layers and that the numerical solution by the Galerkin method on uniform mesh suffers from the Gibbs phenomenon. Thus, it is convenient to solve the problem adaptively. In this paper, we use a modification of the wavelet-based adaptive method from [6] for a spatial discretization, because it has several advantages, namely:

- The adaptivity in the context of a wavelet discretization is simple. It consists in keeping the large wavelet coefficients and discarding the smaller ones.
- The algorithm is asymptotically optimal. It means that the number of floating point operations depends linearly on the number of nonzero wavelet coefficients.

- The condition numbers of stiffness matrices are uniformly bounded.

The paper is organized as follows: In Section 2 we use the backward Euler formula for a discretization in time and derive a variational formulation for the given time level. We define an equivalent l^2 -problem and propose iterations for solving this problem in Section 3. In Section 4 we introduce an implementable version of the iterations and discuss computational issues. Numerical examples are presented in Section 5.

1 Discretization

Let $h > 0$, $t_k := kh$ and u^k be an approximation of $u(\cdot, t_k)$. For simplicity, we use the backward Euler method for a discretization in time:

$$\frac{u^{k+1} - u^k}{\tau} = \mu \Delta u^{k+1} - b \operatorname{div} u^{k+1} - c u^{k+1} + f.$$

Then the variational formulation of our problem at the time level t_k is

$$a_k(u^{k+1}, v) = f^k(v), \quad v \in H_0^1(\Omega), \quad (2)$$

where a continuous bilinear form $a_k : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ and $f^k \in H_0^{-1}(\Omega)$ are given by

$$a_k(u, v) := \mu \int_{\Omega} \nabla u \cdot \nabla v \, dx + b \int_{\Omega} v \operatorname{div} u \, dx + \left(\frac{1}{\tau} + c \right) \int_{\Omega} uv \, dx$$

$$f^k(v) := \frac{1}{\tau} \int_{\Omega} u^k v \, dx + \int_{\Omega} f v \, dx,$$

where $H_0^1(\Omega)$ denotes the subspace of all functions from a Sobolev space $H^1(\Omega)$ with zero traces on $\partial\Omega$, $H_0^{-1}(\Omega)$ denotes its dual. In the sequel, we solve (2) by wavelet-based method. For this reason we propose the definition of a wavelet basis of Sobolev space $H_0^s(\Omega)$.

Family $\Psi := \{\psi_{\lambda}, \lambda = (j, k) \in \mathcal{J}\}$ for an infinite set $\mathcal{J} = \mathcal{J}_{\Phi} \cup \mathcal{J}_{\Psi}$, $\#\mathcal{J}_{\Phi} < \infty$, is called the *wavelet basis* of $H_0^1(\Omega)$, if

- i) Ψ is a Riesz basis of $H_0^1(\Omega)$, that means Ψ generates $H_0^1(\Omega)$ and there exist constants $c, C \in (0, \infty)$ such that for all $\mathbf{b} := \{b_{\lambda}\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J})$, where $\lambda = (j, k)$ and $|\lambda| = j$ denotes the level, holds

$$c \|\mathbf{b}\| \leq \left\| \sum_{\lambda \in \mathcal{J}} b_{\lambda} 2^{-|\lambda|} \psi_{\lambda} \right\|_{H^s(\Omega)} \leq C \|\mathbf{b}\|.$$

- ii) Functions are local in the sense that $\operatorname{diam}(\Omega_{\lambda}) \leq C 2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, where Ω_{λ} is support of ψ_{λ} .

- iii) Functions have cancellation properties of the order m , i.e.

$$|\langle v, \psi_{\lambda} \rangle| \leq 2^{-m|\lambda|} |v|_{H^m(\Omega_{\lambda})}, \quad \lambda \in \mathcal{J}_{\Psi}, \quad v \in H_0^1(\Omega),$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(\Omega)$ inner product.

In this paper, Ψ will be a cubic spline wavelet basis adapted to homogeneous Dirichlet boundary conditions from [1] that has a cancellation property of the order six. The basis functions are local and also their duals are local. Its structure is

$$\Psi = \Phi_4 \cup \bigcup_{j=4}^{\infty} \Psi_j, \quad (3)$$

where Φ_4 contains scaling functions and Ψ_j is formed by wavelets on the level j . Also the smoothness of basis functions is an important property. In this case, the Sobolev exponent of smoothness is 2.5.

2 Wavelet Method

In this section we propose a method for solving (2) based on wavelets. Let \mathbf{D}_k be a diagonal matrix with diagonal elements $\sqrt{a_k(\psi_\lambda, \psi_\lambda)}$. Then $\mathbf{D}_k^{-1}\Psi$ is a wavelet basis in $H_0^1(\Omega)$ and the equation (2) can be reformulated as an equivalent biinfinite matrix equation

$$\mathbf{A}^k \mathbf{u}^k = \mathbf{f}^k, \quad (4)$$

where $\mathbf{A}^k = \mathbf{D}_k^{-1} a_k(\Psi, \Psi) \mathbf{D}_k^{-1}$ is a diagonally preconditioned stiffness matrix, $u^k = (\mathbf{u}^k)^T \mathbf{D}_k^{-1} \Psi$ and $\mathbf{f}^k = \mathbf{D}_k^{-1} f^k(\Psi)$.

Then, u^k solves (2) if and only if \mathbf{u}^k solves the matrix equation (4). Moreover, the matrix \mathbf{A}^k satisfies

$$\text{cond} \mathbf{A}^k \leq C < +\infty. \quad (5)$$

While the classical adaptive methods use refining and derefining a given mesh according to a-posteriori local error indicators, the wavelet approach is somewhat different and follows a paradigm which comprises the following steps:

1. One starts with a variational formulation but instead of turning to a finite dimensional approximation, using the suitable wavelet basis the continuous problem is transformed into an infinite-dimensional l^2 -problem.
2. One then tries to devise convergent iterations for the l^2 -problem.
3. Finally, one derives a practical version of this idealized iteration. All infinite-dimensional quantities have to be replaced by finitely supported ones and the routine for the application of the biinfinite-dimensional matrix \mathbf{A} approximately have to be designed.

We solve the discrete infinite-dimensional problem (4) approximately. For notational simplicity we omit the index k in this section. Matrix $\mathbf{A} := \mathbf{A}^k$ is not symmetric positive definite, but one can obtain a symmetric positive definite formulation by squaring: $\mathbf{L} := \mathbf{A}^T \mathbf{A}$, $\mathbf{g} := \mathbf{A}^T \mathbf{f}$. Then (4) is equivalent to $\mathbf{L}\mathbf{u} = \mathbf{g}$. There exists some relaxation weight ω with

$$\|\mathbf{I} - \omega \mathbf{L}\| \leq \rho < 1. \quad (6)$$

Thus simple iterations of the form

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \omega(\mathbf{g} - \mathbf{L}\mathbf{u}^n) \quad (7)$$

converge. Neither we can evaluate the generally infinite array \mathbf{g} exactly, nor we can compute $\mathbf{L}\mathbf{u}$, even when \mathbf{u} has a finite support. Thus, we need to approximate \mathbf{g} and product $\mathbf{L}\mathbf{u}$ with some given precision.

3 Adaptive Wavelet Scheme

We use the following implementable version of the ideal iteration (7).

SOLVE $[\mathbf{L}, \mathbf{g}, \varepsilon] \rightarrow \mathbf{u}_\varepsilon$

Let ω satisfy (6) and $K \in \mathbb{N}$ be fixed such that $\rho^K < 1/10$.

Set $j := 0$, $\mathbf{u}_0 := 0$, $\varepsilon_0 := \|\mathbf{L}^{-1}\| \|\mathbf{g}\|$.

While $\varepsilon_j > \varepsilon$ do

$j := j + 1$, $\varepsilon_j := 10\rho^K \varepsilon_{j-1}$, $\mathbf{g}_j := \mathbf{RHS}[\mathbf{g}, \frac{\varepsilon_j}{20\omega K}]$, $\mathbf{z}_0 := \mathbf{u}_{j-1}$,

$l := l + 1$, $\mathbf{res}_l := 0$,

If $l \leq K$ and $\|\mathbf{res}_l\| > (\|\mathbf{L}^{-1}\| - \frac{1}{10\omega K}) \varepsilon_j$ do

$l := l + 1$,

$\mathbf{res}_l := \mathbf{g}_j - \mathbf{APPLY}[\mathbf{L}, \mathbf{z}_{l-1}, \frac{\varepsilon_j}{20\omega K}]$,

$\mathbf{z}_l := \mathbf{z}_{l-1} + \omega \mathbf{res}_l$,

end if,

$\mathbf{u}_j := \mathbf{COARSE}[\mathbf{z}_l, 0.7\varepsilon_j]$,

end while,

$\mathbf{u}_\varepsilon := \mathbf{u}_j$.

Let us comment particular subroutines. The subroutine **COARSE** computes a vector \mathbf{v}_ε which is close to the vector of \mathbf{v}^N of N -largest coefficients of \mathbf{v} for which $\|\mathbf{v} - \mathbf{v}^N\| < \varepsilon$. Since sorting of all elements of \mathbf{v} requires $\mathcal{O}(M \log M)$ operations, where M is the length of \mathbf{v} , the procedure **COARSE** uses so called binning [8].

COARSE $[\mathbf{v}, \varepsilon] \rightarrow \mathbf{v}_\varepsilon$

1. Set $q := \left\lceil \log \left((\#\text{supp } \mathbf{v})^{1/2} \|\mathbf{v}\|_{l^2} / \varepsilon \right) \right\rceil$.

2. Regroup the elements of \mathbf{v} into the sets B_0, \dots, B_q , where $\mathbf{v}_\lambda \in B_i$ if and only if

$$2^{-(i+1)} \|\mathbf{v}\|_{l^2} < |\mathbf{v}_\lambda| \leq 2^{-i} \|\mathbf{v}\|_{l^2}, \quad 0 \leq i < q. \quad (8)$$

Possible remaining elements are put into the set B_q .

3. Create \mathbf{v}_ε by collecting nonzero entries from B_0 and when it is exhausted from B_1 and so forth until

$$\|\mathbf{v} - \mathbf{v}_\varepsilon\| \leq \varepsilon. \quad (9)$$

The subroutine

$$\mathbf{RHS}[\mathbf{g}, \varepsilon] \rightarrow \mathbf{g}_\varepsilon \quad (10)$$

approximates the right-hand side \mathbf{g} by the finitely supported vector \mathbf{g}_ε such that

$$\|\mathbf{g} - \mathbf{g}_\varepsilon\| \leq \varepsilon. \quad (11)$$

It can be realized by computing in a preprocessing step a highly accurate approximation to \mathbf{g} in the dual basis along with the corresponding coefficients and then applying of **COARSE** to this finitely supported array of coefficients.

In [2], we proposed the improved matrix-vector multiplication in the context of adaptive wavelet methods. Unlike [3, 9], we are not searching for 2^k greatest vector entries in absolute value, but instead we trace actual decay of matrix and vector entries. Let us denote

$$(\mathbf{L}_k)_{\lambda, \lambda'} := \begin{cases} \mathbf{L}_{\lambda, \lambda'}, & \left| |\lambda| - |\lambda'| \right| \leq k, \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

and

$$S_{\mathbf{L}_k} := \max \left| (\mathbf{L}_k)_{\lambda, \lambda'} \right|, \lambda, \lambda' \in \mathcal{J} \quad (13)$$

for $k \geq 0$, and \mathbf{v}_k contains all vector entries greater than a given tolerance divided by $S_{\mathbf{L}_k}$. Then we can compute

$$\mathbf{L}\mathbf{v} \approx \mathbf{w}_K := \sum_{k=0}^K \mathbf{L}_k \mathbf{v}_k.$$

We can choose K such that

$$\|\mathbf{w}_K - \mathbf{L}\mathbf{v}\| \leq \varepsilon. \quad (14)$$

In [2], it was shown that the number of floating point operations needed for computation of the approximate product $\mathbf{L}\mathbf{v}$, depends linearly on the number of nonzero elements of \mathbf{w}_K .

Theorem 1. *Under the above assumptions, for any $\varepsilon > 0$, the approximations \mathbf{u}_ε produced by SOLVE satisfy*

$$\|u - \mathbf{u}_\varepsilon^T \mathbf{D}^{-1} \Psi\| \lesssim \varepsilon \quad (15)$$

and

$$\#flops \leq C \# \text{supp } \mathbf{u}_\varepsilon, \quad (16)$$

where the constant C does not depend on ε .

Proof. Since the used subroutines are asymptotically optimal, the asymptotical optimality of SOLVE can be proven by similar way as in [6]. \square

As an alternative to the Richardson iterations the steepest descent approach was studied in [7].

4 Numerical examples

In this section, we present two numerical examples.

Example 1. *Let us consider the equation*

$$\mu u'' + u' - u = 1, \quad (17)$$

with homogeneous Dirichlet boundary conditions

$$u(0) = u(1) = 0. \quad (18)$$

It is known that the exact solution is given by

$$u(x) = \frac{(e^l - 1)e^{kx} - (1 - e^k)e^{lx}}{e^k - e^l} - 1, \quad (19)$$

where

$$k = \frac{-1 + \sqrt{1 + 4\mu}}{2\mu}, \quad l = \frac{-1 - \sqrt{1 + 4\mu}}{2\mu}. \quad (20)$$

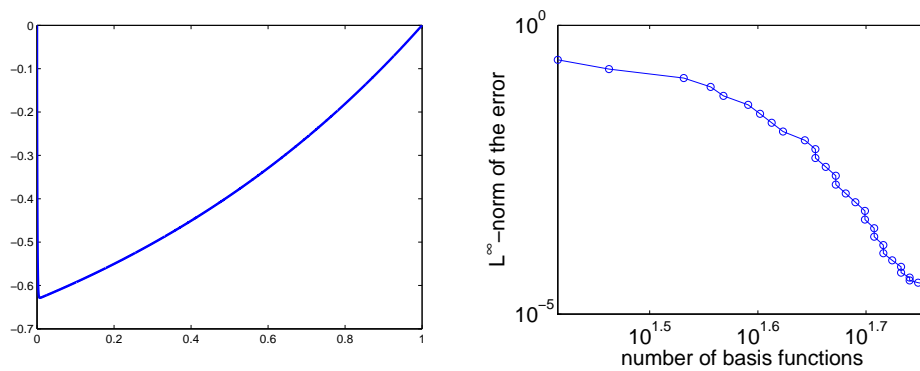
The condition numbers of stiffness matrices \mathbf{A}^s for this equation corresponding to the multi-scale wavelet basis with s levels of wavelets are displayed in Table 1. It can be seen that they are indeed uniformly bounded.

In this case a boundary layer occurs near the point 0. The graph of u for $\mu = 10^{-3}$ is displayed at Figure 1. We solved this equation by an adaptive wavelet scheme from Section 3. Convergence history is shown in Figure 1. Only 54 basis function were used for achieving the error $3.27 \cdot 10^{-5}$ in the L^∞ -norm.

Tab. 1. The condition numbers of stiffness matrices \mathbf{A}^s corresponding to the multi-scale wavelet basis with s levels of wavelets

s	cond \mathbf{A}^s
1	75.1
2	96.7
3	101.2
4	101.2
5	101.8
6	101.8
7	101.8
8	101.8

Source: Own



Source: Own

Fig. 1. The graph of the exact solution (left) and convergence history (right) for Example 1. for $\mu = 10^{-3}$

Example 2. Let us consider the equation

$$\frac{\partial u}{\partial t} = \mu u'' + u' - e^t, \quad (21)$$

with initial and boundary conditions

$$u(x, 0) = u(x) \quad \text{for } x \in [0, 1], \quad u(0, t) = u(1, t) = 0 \quad \text{for } t \in [0, 1], \quad (22)$$

where $u(x)$ is given by (19). Then the exact solution is

$$u(x, t) = u(x) e^t. \quad (23)$$

For $\mu = 10^{-3}$ and $\tau = 0.05$ we need only 355 basis functions to obtain an error less than 10^{-3} at the time $T = 1$.

Conclusion

In this paper we proposed an adaptive wavelet-based method for numerical resolution of the convection-diffusion equation and we briefly reviewed some aspects of a wavelet discretization in connection with our numerical scheme. Computational details and theoretical results one can find in [1, 2, 4, 5, 6].

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ADAPTIVNÍ WAVELETOVÉ SCHÉMA PRO KONVEKTIVNĚ-DIFŮZNÍ ROVNICI

Článek se zabývá teoretickými a výpočetními aspekty numerického řešení konvektivně-difúzních rovnic. Použijeme implicitní schéma pro časovou diskretizaci a adaptivní waveletovou metodu pro prostorovou diskretizaci. Použijeme dobře podmíněnou kubickou spline-waveletovou bázi a metodu pro přibližné násobení waveletové matice tuhosti s vektorem, které jsme nedávno navrhli. Budou prezentovány teoretické výhody našeho schématu a také numerické příklady.

DAS ADAPTIVE WAVELET-SCHEMA FÜR KONVEKTIV-DIFFUSE GLEICHUNGEN

Dieser Artikel befasst sich mit theoretischen und rechnerischen Aspekten der numerischen Lösung konvektiv-diffuser Gleichungen. Dazu verwenden wir ein implizites Schema für die zeitliche Diskretisierung und die adaptive Wavelet-Methode für eine räumliche Diskretisierung. Wir benutzen die gut bedingte kubische Spline-Wavelet-Basis und -Methode für eine annähernde Multiplikation der Zähigkeits-Wavelet-Matrix mit einem Vektor, den wir vor nicht langer Zeit entworfen haben. Es werden theoretische Vorteile unseres Schemas und auch numerische Beispiele vorgestellt.

ADAPTACYJNY SCHEMAT FALKOWY DLA RÓWNANIA KONWEKCYJNO-DYFUZYJNEGO

W artykule przedstawiono teoretyczne i obliczeniowe aspekty numerycznego rozwiązywania równań konwekcyjno-dyfuzyjnych. Zastosowano schemat niejawny (dyskretny) dla czasowej dyskretyzacji oraz adaptacyjną metodę falkową (waveletową) dla dyskretyzacji przestrzennej. Wykorzystujemy dobrze uwarunkowaną kubiczną bazę splajno-falkową oraz metodę przybliżonego mnożenia falkowej macierzy sztywności z wektorem, które niedawno zaproponowaliśmy. Zostały zaprezentowane teoretyczne zalety naszego schematu oraz przykłady numeryczne.