ON A SPARSE REPRESENTATION OF LAPLACIAN

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Abstract

One of the most important part of adaptive wavelet methods is an efficient approximate multiplication of stiffness matrices with vectors in wavelet coordinates. Although there are known algorithms to perform it in linear complexity, the application of them is relatively time consuming and its implementation is very difficult. Therefore, it is necessary to develop a well-conditioned wavelet basis with respect to which both the mass and stiffness matrices are sparse in the sense that the number of nonzero elements in any column is bounded by a constant. Then, matrix-vector multiplication can be performed exactly with linear complexity. We present here a wavelet basis on the interval with respect to which both the mass and stiffness matrices corresponding to the one-dimensional Laplacian are sparse. Consequently, the stiffness matrix corresponding to the n-dimensional Laplacian in tensor product wavelet basis is also sparse. Moreover, the constructed basis has an excellent condition number. In this contribution, we shortly review this construction and show several numerical tests.

Keywords: Wavelet; Hermite cubic splines; sparse representations.

Introduction

A general concept for solving of operator equations by means of wavelets was proposed by A. Cohen, W. Dahmen and R. DeVore in [2, 3]. The aim of this concept consists in the approximation of the unknown solution u which should correspond to the best N-term approximation, and the associated computational work should be proportional to the number of unknowns. It consists of the following steps: transformation of the variational formulation into the well-conditioned infinite-dimensional problem in the space l^2, finding of the convergent iteration process for the l^2 problem and finally derivation of its computable version. The essential step to achieve this goal is an efficient approximate multiplication of quasi-sparse wavelet matrices with vectors.

In [2], authors exploited an off-diagonal decay of entries of the wavelet stiffness matrices and designed a numerical routine **APPLY** which approximates the exact matrix-vector product with the desired tolerance ε and that has linear computational complexity, up to sorting operations. The idea of **APPLY** is following: To truncate $\mathbf{A}$ in scale by zeroing $a_{i,j}$ whenever $\delta(i,j) > k$ (δ represents the level difference of two functions in the wavelet expansion) and denote the resulting matrix by $\mathbf{A_k}$. At the same time to sort vector entries $\mathbf{v}$ with respect to the size of their absolute values. One obtains $\mathbf{v_k}$ by retaining 2^k greatest coefficients in absolute values of $\mathbf{v}$ and setting all other equal to zero. The maximum value of k should be determined to reach a desired accuracy of approximation. Then one computes an approximation of $\mathbf{Av}$ by

$$\mathbf{w} := \mathbf{A_k v_0} + \mathbf{A_{k-1}}(\mathbf{v_1} - \mathbf{v_0}) + \ldots + \mathbf{A_0}(\mathbf{v_k} - \mathbf{v_{k-1}}) \tag{1}$$

with the aim to balance both accuracy and computational complexity at the same time. In [6], binning and approximate sorting strategy was used to eliminate these sorting costs and then an asymptotically optimal algorithm was obtained. The idea is following: Reorder the elements of $\mathbf{v}$ into the sets $V_0, \ldots, V_q$, where $v_\lambda \in V_i$ if and only if

$$2^{-i-1} \|\mathbf{v}\|_{l^2} < v_\lambda < 2^{-i} \|\mathbf{v}\|_{l^2}, \quad 0 \le i < q.$$

Eventual remaining elements are put into the set V_q. And subsequently to generate vectors $\mathbf{v_k}$ by successively extracting 2^k elements from $\bigcup_i V_i$, starting from V_0 and when it is empty continuing with V_1 and so forth. Finally, the scheme (1) is applied. Further improvements of this scheme were proposed in [1, 4].

The described scheme was shown to possess the best possible rate of convergence in linear complexity and since tensor product wavelets are applied, this rate is independent of the space dimension [4]. Although it has optimal computational complexity, the application of the **APPLY** routine is relatively time consuming and moreover it is not easy to implement it efficiently. Therefore, it is necessary to develop a well-conditioned wavelet basis with respect to which both the mass and stiffness matrix are sparse. This means that the number of nonzero elements in any column as well as the condition number of stiffness matrices are bounded independent of the matrix size which is not generally true using a wavelet discretization. Then, a matrix-vector multiplication can be performed exactly with linear complexity.

The remainder of this paper is organized as follows. In the next section, we present a construction of a wavelet basis on the interval based on Hermite multiwavelets with respect to which both the mass and stiffness matrices corresponding to the one-dimensional Laplacian are sparse and very well-conditioned. We also identify several free parameters in the construction of the second two wavelets which can be used to improve properties of a constructed basis. In the last section, we present several numerical tests to compare a proposed basis with the basis proposed in [5].

1 Hermite Multiwavelets

We start with Hermite cubic splines as the primal scaling bases on the interval. They are defined by

$$\phi_1(x) = \begin{cases} (x+1)^2(1-2x) & -1 \le x \le 0 \\ (1-x)^2(2x+1) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}, \quad \phi_2(x) = \begin{cases} (x+1)^2 x & -1 \le x \le 0 \\ (1-x)^2 x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

For $n \geq 1$, let V_n be the space of piecewise cubic splines $v \in C^1(0,1) \cap C[0,1]$ for which $v(0) = v(1) = 0$. The dimension of V_n is 2^{n+1} and the set

$$\Phi_n := \{\phi_1(2^n x - j) : j = 1, ..., 2^n - 1\} \cup \{\phi_2(2^n x - j)|_{[0,1]} : j = 0, ..., 2^n\}$$

is the basis for V_n. Let W_n be the complement of V_n in V_{n+1} then we have the following decomposition of the space $H_0^1(0,1)$

$$H_0^1(0,1) = V_1 + W_1 + W_2 + W_3 \ldots.$$

We follow the construction proposed in [5] and construct four wavelets. Wavelets from the space W_{n+1} are orthogonal to the scaling functions from the space V_n for $n \geq 1$. This property ensures that both the mass and stiffness matrix corresponding to the one-dimensional Laplacian have at most three wavelet blocks of nonzero elements in any column and then the number of nonzero elements in any column is bounded independent of the matrix size. The first two wavelets have supports in $[-1,1]$ and are uniquely determined by the above orthogonality condition and by imposing that the first one is odd and the second one is even. The second two wavelets have supports in $[-2,2]$. And we impose on them the orthogonality condition and again one of them should be odd and the second one even.

Then the space W_n is defined by

$$\begin{aligned} \Psi_n := & \left\{\psi_1(2^n x - 2j - 1), \psi_2(2^n x - 2j - 1) : j = 0, ..., 2^{n-1} - 1\right\} \\ & \cup \left\{\psi_3(2^n x - 2j) : j = 1, ..., 2^{n-1} - 1\right\} \cup \left\{\psi_4(2^n x - 2j)|_{[0,1]} : j = 0, ..., 2^{n-1}\right\}. \end{aligned}$$

There remain several free parameters in the construction of the second two wavelets. In [5], these parameters were used to prescribe the orthogonality to the first two wavelets. We try to use these parameters to obtain better conditioned basis and alternatively also more sparse stiffness matrices. Some preliminary results are presented in the next section.

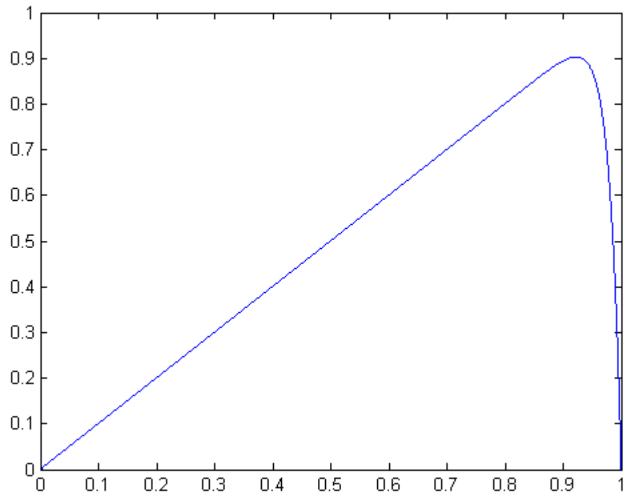
2 Numerical Experiments

We will limit ourselves to the equation $-pu'' + qu = f$ with the Dirichlet boundary conditions $u(0) = u(1) = 0$ and with positive constant coefficients. The corresponding Galerkin approximation problem is the following: Find $u_n = \sum_{i=1}^{2^{n+2}} c_i v_i$ such that

$$\int_0^1 p u_n' v' + q u_n v \, dx = \int_0^1 f v \, dx \qquad \forall v \in V_n.$$

To solve arising system of equations, we use the standard wavelet diagonal preconditioning and then apply the conjugate gradient method. The iterations are terminated if the difference of two consecutive iterations is less than $10^{-n-2}/\mathrm{cond}(A_n)$, where $\mathrm{cond}(A_n)$, denotes the condition number of the corresponding stiffness matrix.

First, we solve the above problem with $p = q = 1$ and with $p = 1,\ q = 0$. In both examples, the exact solution is $u = x(1 - e^{50x-50})$ which exhibits a steep gradient near the point 1. See Figure 1. Obtained results are summarized in Table 1. In this table, NIT represents the number of iterations, NEW denotes new wavelets, and finally DS denotes wavelets proposed in [5]. The achieved approximation error was the same in all cases.



Source: Own

***Fig. 1.** The exact solution*

***Tab. 1.** Obtained results*

		$p=1,\ q=0$		$p=q=1$	
n	$\|\|u_n-u\|\|_{L_2}$	NIT DS	NIT NEW	NIT DS	NIT NEW
1	6.0e-02	5	4	5	4
2	1.7e-02	7	6	7	6
3	2.6e-03	9	7	8	7
4	2.7e-04	10	9	10	9
5	2.3e-05	13	10	12	11
6	1.8e-06	14	12	14	12
7	1.2e-07	16	13	15	13
8	8.3e-09	17	15	16	15
9	5.3e-10	19	16	19	16
10	3.4e-11	19	17	19	18

Source: Own

Conclusion

The presented numerical results affirm that the proposed wavelet basis is better conditioned then the wavelet basis proposed in [5] which resulted in slightly smaller numbers of required iterations. Moreover, the numbers of nonzero elements in stiffness matrices corresponding to Poisson equation were substantially smaller. We believe that it will be possible to further improve the proposed construction.

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O Řídké reprezentaci Laplaciánu

Jednou z nejdůležitějších částí waveletových adaptivních metod je efektivní přibližné násobení matice tuhosti s vektory ve waveletových souřadnicích. Přestože jsou známy algoritmy pro přibližné násobení s lineární složitostí, jejich aplikace je relativně časově náročná a jejich implementace velice obtížná. Proto je důležité vyvíjet dobře podmíněné waveletové báze, pro které jsou jak matice hmotnosti, tak matice tuhosti řídké ve smyslu, že počet nenulových prvků v libovolném sloupci je omezený konstantou. Potom je totiž možné násobit matici s vektorem přesně s lineární přesností. V tomto příspěvku prezentujeme waveletovou bázi adaptovanou na interval, vzhledem ke které jsou jak matice hmotnosti, tak matice tuhosti odpovídající jednodimenzionálnímu Laplaciánu řídké. Následně je rovněž matice tuhosti odpovídající n - dimenzionálnímu Laplaciánu ve waveletové bázi, která vznikne tenzorovým součinem jednodimenzionálních bází, řídká. Navíc jsou konstruované báze velmi dobře podmíněné. V tomto příspěvku krátce představíme tuto konstrukci a ukážeme několik numerických testů.

Über die seltene Repräsentation von Laplazian

Eine der der wichtigsten Teile der adaptiven Wavelet-Methoden besteht in der annähernden Multiplikation der Zähigkeitsmatrix mit Vektoren in den Wavelet-Koordinaten. Obschon Algorithmen fr eine annähernde Multiplikation mit linearer Komplexität bekannt sind, ist deren Anwendung zeitlich gesehen relativ aufwändig und ihre Implementierung sehr beschwerlich. Daher ist es wichtig, eine gut bedingte Wavelet-Basis zu entwickeln, für welche sowohl die Massen- als auch die Zähigkeitsmatrix insofern selten sind, als die Anzahl der Null-Elemente in einer beliebigen Kolumne durch die Konstante begrenzt ist. Hernach ist es nämlich möglich, die Matrix mit dem Vektor genau mit linearer Genauigkeit zu multiplizieren. In diesem Beitrag präsentieren wir die ans Intervall angepasste Wavelet-Basis, hinsichtlich derer sowohl die Massen- als auch die dem eindimensionalen Laplazian entsprechende Zähigkeitsmatrix selten sind. Auch die dem n-dimensionalen Laplazian in der Wavelet-Basis entsprechende Zähigkeitsmatrix, die durch das Tensorprodukt eindimensionaler Basen entsteht, ist selten. Darüber hinaus werden die konstruierten Basen sehr gut bedingt. In diesem Beitrag stellen wir kurz diese Konstruktion vor und zeigen einige numerische Tests.

O rzadkiej reprezentacji laplasjanu

Jedną z najważniejszych części falkowych metod adaptacyjnych jest efektywne przybliżone mnożenie macierzy sztywności z wektorami we współrzędnych waveletowych. Chociaż znane są algorytmy służące do przybliżonego mnożenia o złożoności liniowej, ich stosowanie jest stosunkowo czasochłonne a ich wdrażanie bardzo trudne. Dlatego ważne jest opracowywanie dobrze uwarunkowanej bazy falkowej, dla której macierze masy, jak również macierze sztywności są rzadkie, to znaczy liczba elementów niezerowych w dowolnej kolumnie nie jest ograniczona stałą. Wówczas bowiem można macierz z wektorem mnożyć z dokładnością liniową. W niniejszym artykule zaprezentowano bazę falkową zaadaptowaną do interwału, w stosunku do której macierze masy, jak również macierze sztywności odpowiadające jednowymiarowemu laplasjanowi są rzadkie. Następnie także macierz sztywności odpowiadająca n-wymiarowemu lapsjanowi w bazie falkowej (waveletowej), która powstaje w wyniku iloczynu tensorowego baz jednowymiarowych, jest rzadka. Ponadto konstruowane bazy są bardzo dobrze uwarunkowane. W niniejszym artykule krótko przedstawiono taką konstrukcję oraz pokazano kilka testów numerycznych.