MULTIWAVELETS BASED ON HERMITE CUBIC SPLINES

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Abstract

First multiwavelets have appeared around the early 1990s. The basic idea behind multiwavelets is simple: to replace the single scaling function ϕ by the multiscaling function $\mathbf{\Phi}$ to have some additional desired properties. It seems to be an interesting trade off because multiwavelets provide higher order approximation with shorter support than single scaling function. Moreover, it is possible to have both symmetric and orthogonal multiwavelets while this is not possible for single wavelets. In recent years, several simple constructions of wavelet bases based on Hermite cubic splines were proposed. In this contribution, we shortly review these constructions, use these wavelets to solve numerically differential equations, and compare their performance.

Keywords: Wavelet; Hermite cubic splines; elliptic differential equations.

Introduction

A vector-valued function

$$\mathbf{\Phi}(x) = (\phi_1(x), \phi_2(x), \dots, \phi_r(x))^T, \qquad \phi_1(x), \phi_2(x), \dots, \phi_r(x) \in L^2(\mathbb{R})$$

will be called a multiscaling function if it satisfies the following refinement equation

$$\mathbf{\Phi}(x) = \sum_{k=s}^{t} \mathbf{H_k} \mathbf{\Phi}(ax-k),$$

where $s, t \in \mathbb{Z}$, $s < t$, $\mathbf{H_k}$ are some $r \times r$ matrices, $a \in \mathbb{N}$ and $a \geq 2$. The multiscaling function generates a multiresolution analysis of $L^2(\mathbb{R})$ in a similar way as for scalar wavelets. Subspaces V_j are defined

$$V_j = \text{clos}_{L^2(\mathbb{R})} \{ \phi_{l,j,k} = a^{j/2} \phi_l (a^j x - k) : 1 \leq l \leq r, k \in \mathbb{Z} \}, \qquad j \in \mathbb{Z}.$$

Then W_j, $j \in \mathbb{Z}$ denotes the orthogonal complementary subspaces of V_j in V_{j+1} and the vector-valued function

$$\boldsymbol{\Psi}(x) = (\psi_1(x), \psi_2(x), \ldots, \psi_{(a-1)r}(x))^T, \qquad \psi_1(x), \psi_2(x), \ldots, \psi_{(a-1)r}(x) \in L^2(\mathbb{R}).$$

Therefore there exist matrices $\mathbf{Q_k}$ such that

$$\boldsymbol{\Psi}(x) = \sum_{k=s}^{t} \mathbf{Q_k} \boldsymbol{\Phi}(ax-k),$$

and fast decomposition and reconstruction algorithms can be constructed like for scalar wavelets. Moreover, multiwavelets usually have shorter support than corresponding single wavelets, it is possible to construct both symmetric and orthogonal multiwavelets while this is not possible for single wavelets, and finally, it is possible to have a basis formed by functions with different orders (see Example 1). However, multiwavelets have also some disadvantages: the discrete multiwavelet transform usually requires preprocessing and postprocessing, and their construction is also more complicated. The reason, why preprocessing is necessary, is in the fact that for scalar wavelets, we have

$$2^{-n/2} f\left(2^{-n} k\right) \approx \int f(x)\, \phi_{n,k}(x)\, dx$$

and this does not hold in general for multiwavelets. Multiwavelets which do not require preprocessing are for instance the so called balanced multiwavelets. They were constructed in [7, 8]. For more details on multiwavelets, we refer to [6].

Example 1 *Simple example of piecewise linear multiwavelets is taken from [1]:*

$$\phi_1(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \textit{otherwise} \end{cases}, \qquad \phi_2(x) = \begin{cases} 2\sqrt{3}\left(x - \frac{1}{2}\right) & 0 \le x \le 1 \\ 0 & \textit{otherwise} \end{cases},$$

$$\psi_1(x) = \begin{cases} 6x-1 & 0 \le x < \frac{1}{2} \\ 6x-5 & \frac{1}{2} \le x \le 1 \\ 0 & \textit{otherwise} \end{cases}, \qquad \psi_2(x) = \begin{cases} 2\sqrt{3}\left(2x - \frac{1}{2}\right) & 0 \le x < \frac{1}{2} \\ -2\sqrt{3}\left(2x - \frac{3}{2}\right) & \frac{1}{2} \le x \le 1 \\ 0 & \textit{otherwise} \end{cases}.$$

The remainder of this paper is organized as follows. In the next section, we review several simple constructions of wavelets based on Hermite cubic splines. Specifically, wavelets with two vanishing wavelet moments proposed in [5], a hierarchical basis based on Hermite cubic splines, wavelets with four vanishing wavelet moments proposed in [2], and wavelets with respect to which both the mass and stiffness matrices corresponding to the one-dimensional Laplacian are sparse [3]. In the last section, we use these wavelets to solve numerically differential equations and compare their performance.

1 Hermite Cubic Spline Wavelets

In the year 2000, W. Dahmen et al. [4] proposed a construction of biorthogonal multiwavelets adapted to the interval $[0,1]$ on the basis of Hermite cubic splines. They started with Hermite cubic splines as the primal scaling bases on $\mathbb{R}$. Then, they constructed dual scaling bases on $\mathbb{R}$ consisting of continuous functions with small supports and with polynomial exactness of order 2. Consequently, they derived primal and dual boundary scaling functions retaining the polynomial exactness. This ensures vanishing moments of the corresponding wavelets. Finally,

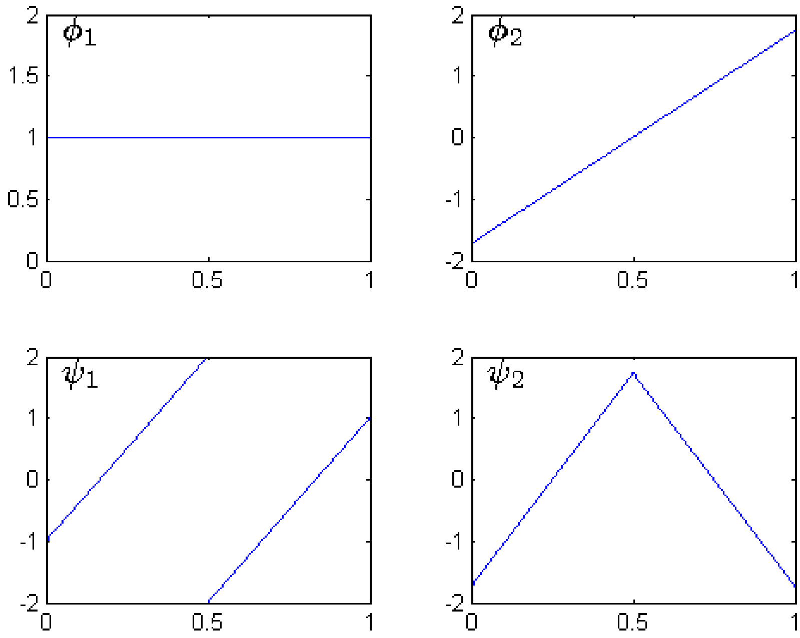

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Fig. 1. *Piecewise linear multiwavelets*

they applied the method of stable completions to construct the corresponding primal and dual multiwavelets on the interval.

These Hermite cubic splines are defined by

$$\phi_1(x) = \begin{cases} (x+1)^2(1-2x) & -1 \le x \le 0 \\ (1-x)^2(2x+1) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}, \quad \phi_2(x) = \begin{cases} (x+1)^2 x & -1 \le x \le 0 \\ (1-x)^2 x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

They possess the following interpolation property:

$$\phi_1(0) = 1, \quad \phi_1'(0) = 0, \quad \phi_2(0) = 0, \quad \phi_2'(0) = 1$$

and then for any function $f \in C^1(\mathbb{R})$

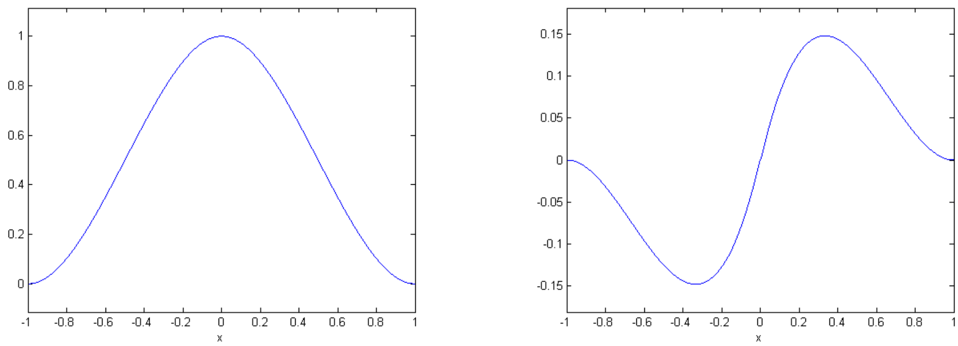
$$u := \sum_{j \in \mathbb{Z}} f(j)\phi_1(x-j) + \sum_{j \in \mathbb{Z}} f'(j)\phi_2(x-j)$$

is a Hermite interpolant to f on $\mathbb{Z}$.

1.1 Wavelets proposed by R. Q. Jia and S. T. Liu

An interesting construction was proposed in [5]. The authors constructed multiwavelets based on Hermite cubic splines with continuous derivatives and with a support on the interval $[-1,1]$. One of the constructed wavelets is symmetric, and the second one is antisymmetric. They also adapted them to the interval $[0,1]$ and their construction of boundary wavelets is very simple unlike the construction from [4]. In comparison with the semi-orthogonal wavelets, the wavelets at different levels are orthogonal with respect to $\langle u', v' \rangle$ instead of $\langle u, v \rangle$. They are given by

$$\psi_1(x) = -2\phi_1(2x+1) + 4\phi_1(2x) - 2\phi_1(2x-1) - 21\phi_2(2x+1) + 21\phi_2(2x-1),$$



ϕ_1 ϕ_2

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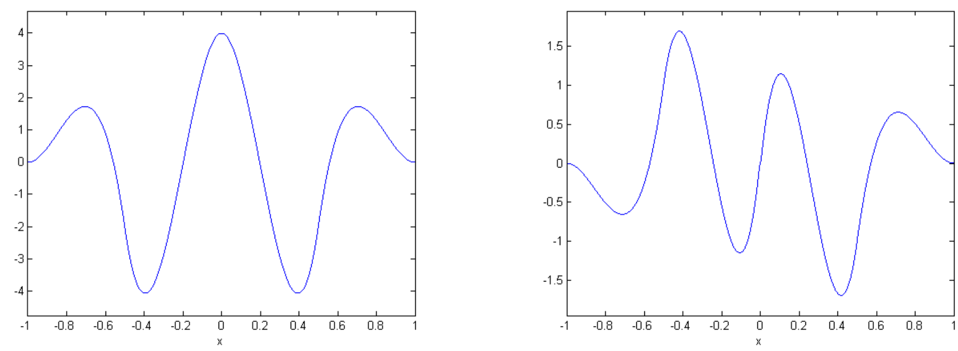
Fig. 2. *The Hermite cubic spline*



ψ_1 ψ_2

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Fig. 3. *Wavelets proposed by R. Q. Jia and S. T. Liu*

$$\psi_2(x) = \phi_1(2x+1) - \phi_1(2x-1) + 9\phi_2(2x+1) + 12\phi_2(2x) + 9\phi_2(2x-1).$$

Both wavelets are supported on $[-1,1]$ and its adaptation to the interval $[0,1]$ is performed by the restriction of the antisymmetric wavelet to the interval $[0,1]$. For $n \geq 1$, let V_n be the space of piecewise cubic splines $v \in C^1(0,1) \cap C[0,1]$ for which $v(0) = v(1) = 0$. The dimension of V_n is 2^{n+1} and the set

$$\Phi_n := \{\phi_1(2^n x - j) : j = 1, ..., 2^n - 1\} \cup \{\phi_2(2^n x - j)|_{[0,1]} : j = 0, ..., 2^n\}$$

is the basis for V_n. Let W_n be the complement of V_n in V_{n+1} with the basis defined by

$$\Psi_n := \{\psi_1(2^n x - j) : j = 1, ..., 2^n - 1\} \cup \{\psi_2(2^n x - j)|_{[0,1]} : j = 0, ..., 2^n\}.$$

Then, we have the following decomposition of $H_0^1(0,1)$

$$H_0^1(0,1) = V_1 + W_1 + W_2 + W_3 \ldots.$$

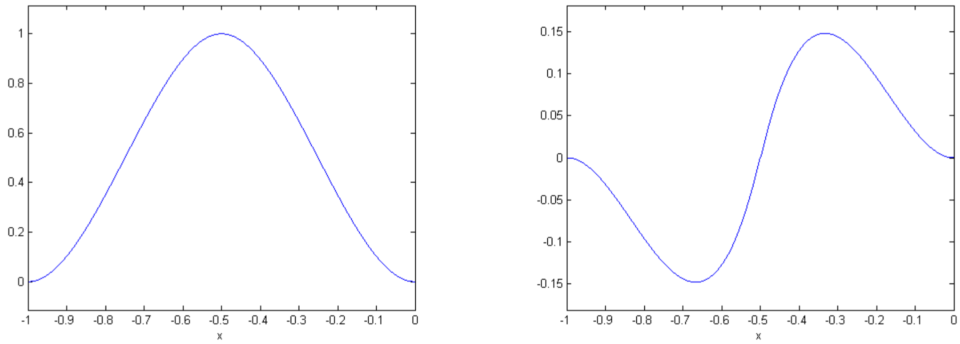
1.2 Hierarchical Basis

The second possibility to define complement wavelet spaces, we use the following hierarchical approach:

$$\psi_1(x) = \phi_1(2x+1) \quad \text{and} \quad \psi_2(x) = \phi_2(2x+1).$$

Both functions are supported on $[-1,0]$ and therefore no boundary functions are necessary. The complement space W_n is then given by

$$\Psi_n := \{\psi_1(2^n x - j) : j = 1, \ldots, 2^n\} \cup \{\psi_2(2^n x - j) : j = 1, \ldots, 2^n\}.$$



ψ_1 ψ_2

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Fig. 4. *Hierarchical wavelets*

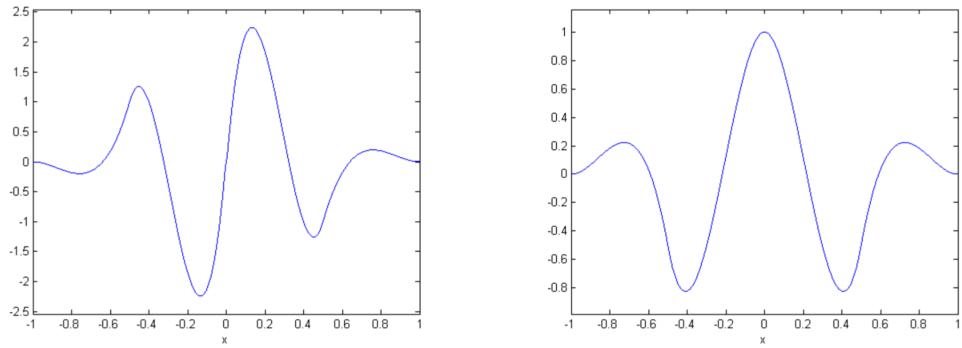
1.3 Wavelets with Moments

The third possibility is to define wavelets to have maximal number of vanishing moments for the given support $[-1,1]$. Wavelets are given then by

$$\psi_1(x) = \phi_1(2x+1) - \phi_1(2x-1) + \frac{39}{7}\phi_2(2x+1) + \frac{132}{7}\phi_2(2x) + \frac{39}{7}\phi_2(2x-1),$$

and

$$\psi_2(x) = -\frac{1}{2}\phi_1(2x+1) + \phi_1(2x) - \frac{1}{2}\phi_1(2x-1) - \frac{15}{4}\phi_2(2x+1) + \frac{15}{4}\phi_2(2x-1).$$



ψ_1 ψ_2

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Fig. 5. *Wavelets with the maximal number of vanishing moments*

Boundary wavelets are constructed to have the same number of vanishing moments as inner wavelets. For more details, see [2]. Again let W_n be the complement of V_n in V_{n+1} with the basis defined by

$$\begin{aligned}\Psi_n \quad := \quad & \{\psi_1(2^n x - j),\, \psi_2(2^n x - j) : j = 1, \ldots, 2^n - 1\} \\ & \cup \psi_3(2^n x - 1)|_{[0,1]} \cup \psi_4(2^n(x-1)+1)|_{[0,1]}.\end{aligned}$$

1.4 Wavelets Proposed by T. J. Dijkema and R. Stevenson

Further construction was proposed in [3]. They first proved that it is not possible to construct continuous piecewise smooth wavelets which have a compact support, form Riesz basis for $L^2(I)$, properly scaled wavelets form the Riesz basis for $H^1(I)$, and with

$$\left\langle \psi'_\lambda, \psi'_\mu \right\rangle = 0, \quad \text{and} \quad \langle \psi_\lambda, \psi_\mu \rangle = 0 \quad \forall \lambda, \mu.$$

Consequently, they constructed cubic Hermite wavelets with the continuous first derivative such that among the primal multiresolution spaces V_j and the dual multiresolution spaces $\tilde{V}_j$ the following relation holds

$$V_j + V_j'' \subset \tilde{V}_{j+1}.$$

As a consequence

$$\left\langle \psi'_\lambda, \psi'_\mu \right\rangle = 0, \quad \langle \psi_\lambda, \psi_\mu \rangle = 0, \quad \forall \lambda, \mu : |\lambda| > |\mu| + 1.$$

Their wavelets are then given by

$$\psi_1(x) = \phi_1(2x+1) - \phi_1(2x-1) + \frac{39}{7}\phi_2(2x+1) + \frac{132}{7}\phi_2(2x) + \frac{39}{7}\phi_2(2x-1),$$

$$\psi_2(x) = -\frac{1}{2}\phi_1(2x+1) + \phi_1(2x) - \frac{1}{2}\phi_1(2x-1) - \frac{15}{4}\phi_2(2x+1) + \frac{15}{4}\phi_2(2x-1),$$

$$\psi_3(x) = \sum_{i=-3}^{3} c_i\phi_1(2x-i) + d_i\phi_2(2x-i), \quad \psi_4(x) = \sum_{i=-3}^{3} e_i\phi_1(2x-i) + f_i\phi_2(2x-i)$$

with

$$c = \left[-\frac{4595}{13728}, \frac{7}{65}, -\frac{18737}{68640}, 1, -\frac{18737}{68640}, \frac{7}{65}, -\frac{4595}{13728}\right],$$

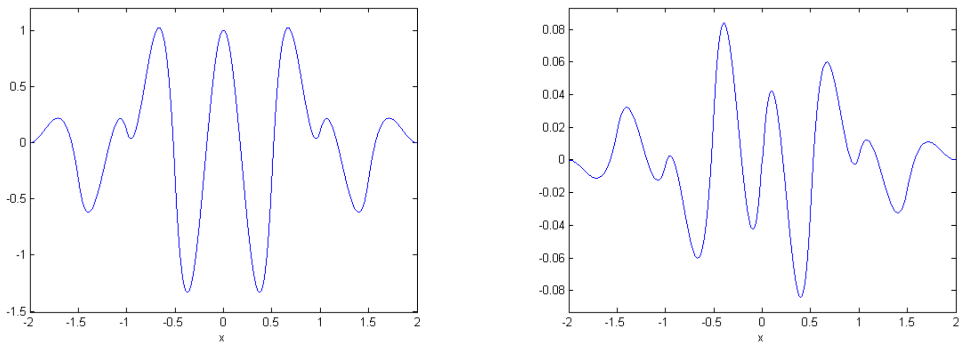
$$d = \left[-\frac{68741}{22880}, -\frac{69}{40}, -\frac{204701}{22880}, 0, \frac{204701}{22880}, \frac{69}{40}, \frac{68741}{22880}\right],$$

$$e = \left[\frac{417}{22880}, -\frac{7}{2340}, \frac{5443}{205920}, 0, -\frac{5443}{205920}, \frac{7}{2340}, -\frac{417}{22880}\right],$$

$$f = \left[\frac{723}{4576}, \frac{1}{8}, \frac{8153}{13728}, \frac{1}{2}, \frac{8153}{13728}, \frac{1}{8}, \frac{723}{4576}\right].$$

Again W_n is the complement of V_n in V_{n+1} with the basis defined by

$$\begin{aligned}\Psi_n \quad := \quad & \left\{\psi_1(2^n x - 2j - 1),\, \psi_2(2^n x - 2j - 1) : j = 0, \ldots, 2^{n-1} - 1\right\} \\ & \cup \left\{\psi_3(2^n x - 2j) : j = 1, \ldots, 2^{n-1} - 1\right\} \cup \left\{\psi_4(2^n x - 2j)|_{[0,1]} : j = 0, \ldots, 2^{n-1}\right\}.\end{aligned}$$



ψ_3 ψ_4

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Fig. 6. *Wavelets proposed by T. J. Dijkema and R. Stevenson*

2 Numerical Experiments

In this section, the wavelets introduced in the previous section are used to solve numeri-cally differential equations. We will restrict ourselves to the equation $-pu'' + qu = f$ with the Dirichlet boundary conditions $u(0) = u(1) = 0$ and with positive constant coefficients. The corresponding Galerkin approximation problem is the following: Find $u_n = \sum_{i=1}^{2^{n+2}} c_i v_i$ such that

$$\int_0^1 p u_n' v' + q u_n v \, dx = \int_0^1 f v \, dx \qquad \forall v \in V_n.$$

By the Lax-Milgram lemma, this approximation problem has the unique solution. We also use the standard wavelet preconditioning consisting in normalizing each basis function with respect to the above bilinear form. To solve the arising system of linear equations, we use the conjugate gradient method. The iterations are terminated if the difference of two consecutive iterations is less than $10^{-n-2}/\mathrm{cond}(A_n)$, where $\mathrm{cond}(A_n)$ denotes the condition number of the corresponding stiffness matrix.

Tab. 1. *Obtained results for the problem with $p = q = 1$.*

		JL		H		M		DS	
n	$\|\|u_n - u\|\|_{L_2}$	NZ	IT	NZ	IT	NZ	IT	NZ	IT
1	6.0e-02	48	4	40	7	54	7	54	5
2	1.7e-02	172	5	128	13	162	11	154	7
3	2.6e-03	552	6	368	20	418	16	410	9
4	2.7e-04	1580	7	976	31	1010	21	986	10
5	2.3e-05	4168	8	2448	41	2306	24	2202	13
6	1.8e-06	10396	10	5904	52	5042	25	4698	14
7	1.2e-07	24936	10	13840	74	10690	28	9754	16
8	8.3e-09	58283	11	31760	84	22194	31	19967	17
9	5.4e-10	135302	12	71696	110	46963	33	41627	19

Source: Own

Tab. 2. *Obtained results for the problem with $p = 1$ and $q = 0$.*

		JL		H		M		DS	
n	$\|\|u_n - u\|\|_{L_2}$	NZ	IT	NZ	IT	NZ	IT	NZ	IT
1	6.0e-02	24	3	36	7	50	7	54	5
2	1.7e-02	56	5	116	12	154	11	154	7
3	2.6e-03	128	6	340	20	406	16	410	8
4	2.7e-04	280	6	916	31	994	20	986	10
5	2.3e-05	592	8	2324	41	2286	24	2202	12
6	1.8e-06	1224	10	5652	52	5018	25	4698	14
7	1.2e-07	2847	10	13332	69	10677	28	9762	15
8	8.3e-09	8818	10	30740	84	22347	30	20019	16
9	5.4e-10	35863	12	69652	110	46613	33	41421	19

Source: Own

First we solve the above problem with $p = q = 1$ and the exact solution $u = x(1 - e^{50x-50})$ which exhibits a steep gradient near the point 1. Results are summarized in Table 1. Then we solve the above problem with $p = 1,\ q = 0$ and with the exact solution $u = x(1 - e^{50x-50})$ which exhibits a steep gradient near the point 1. Results are summarized in Table 2. In both tables, NZ is the number of nonzero elements in stiffness matrices, IT represents the number of iterations, JL denotes wavelets proposed in [5], H denotes hierarchical basis, M denotes wavelets proposed in [2], and finally DS denotes wavelets proposed in [3]. Achieved approximation error was the same for all bases.

Conclusion

Presented results affirm that wavelets proposed in [5] have excellent condition number, and especially for the Poisson equation, the arising stiffness matrices are very sparse. Wavelets proposed in [3] are best suited for general differential equations with constant coefficients. Results of tested hierarchical basis confirm the well known fact that these basis do not form the Riesz basis and therefore they need some additional preconditioning. Concerning the basis proposed in [2], we suppose that it will be better (than tested bases) suited for non-constant differential equations solved adaptively. Presented results also suggest that there is probably some space to improve the condition number of the basis proposed in [2].

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Literature

[1] ALPER, B. K.; ROKHLIN, V.: A fast algorithm for the evaluation of Legendre expansions. *Siam J. Sci. Statist. Comput.*, 12 (1991), pp. 158-179.

[2] ČERNÁ, D.; FINĚK, V.: *Hermite cubic spline wavelets with moments on the interval*. In preparation, 2012.

[3] DIJKEMA, T. J.; STEVENSON, R.: A sparse Laplacian in tensor product wavelet coordinates. *Numer. Math.*, vol. 115, no. 3 (2010), pp. 433-449.

[4] DAHMEN, W.; HAN, B.; JIA, R.-Q.; KUNOTH, A.: Biorthogonal Multiwavelets on the Interval: Cubic Hermite Splines. *CONSTR. APPROX.*, vol. 16, (1998), pp. 221-259.

[5] JIA, R.-Q.; LIU, S.-T.: Wavelet bases of Hermite cubic splines on the interval. *Adv. in Comp. Math.*, 25 (2006), pp. 23-39.

[6] KEINERT, F.: *Wavelets and Multiwavelets*. CHAPMAN & HALL/CRC, 2004.

[7] LEBRUN, J.; VETTERLI, M.: Balanced multiwavelets theory and design. *IEEE Trans. Signal Process.*, 46 (1998), pp. 1119-1125.

[8] LEBRUN, J.; VETTERLI, M.: High-order balanced multiwavelets: theory, factorization, and design. *IEEE Trans. Signal Process.*, 49 (2001), pp. 1918-1930.

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MULTIWAVELETY ZALOŽENÉ NA HERMITOVSKÝCH KUBICKÝCH SPLINECH

První multiwaveletová báze se objevila kolem roku 1990. Základní myšlenka multiwaveletů je jednoduchá: nahradíme jednu škálovou funkci jednou multiškálovou, abychom získali lepší vlastnosti bázových funkcí. Multiwavelety totiž umožňují aproximace vyššího řádu s kratším nosičem než klasické škálové funkce. Navíc je možné zkonstruovat symetrické ortogonální multiwavelety, což pro klasické wavelety není možné. V minulých letech se objevilo několik jednoduchých konstrukcí waveletových bází založených na hermitovských kubických splinech. V tomto příspěvku stručně představíme tyto konstrukce, použijeme tyto wavelety k numerickému řešení diferenciálních rovnic a srovnáme jejich vlastnosti.

AUF KUBISCHEN HERMITE-SPLINES ANGELEGTE MULTIWAVELETS

Die erste Multiwaveletbasis erschien um das Jahr 1990. Der Grundgedanke der Multiwavelets ist einfach: Wir ersetzen eine Skalenfunktion durch eine Multiskalenfunktion, um bessere Eigenschaften von Basenfunktionen zu erhalten. Die Multiwavelets ermöglichen nämlich eine Annäherung höherer Ordnung mit einem kürzeren Träger als die klassischen Skalenfunktionen. Außerdem ist es möglich, symmetrische orthogonale Multiwavelets zu konstruieren, was für die klassischen Wavelets nicht möglich ist. In den vergangenen Jahren erschienen einige einfache Konstruktionen von Wavelet-Basen, die auf kubischen Hermite-Splinen beruhen. In diesem Beitrag stellen wir kurz diese Konstruktionen vor. Wir benutzen die Wavelets zur numerischen Lösung von Differenzialgleichungen und vergleichen deren Eigenschaften.

FALKI WIELOKROTNE OPARTE NA HERMITOWSKICH SPLAJNACH KUBICZNYCH

Pierwsza falka wielokrotna (ang. multiwavelets) pojawiła się około 1990 roku. Podstawowa idea falek wielokrotnych jest prosta: zastępujemy jedną funkcję skalującą jedną wieloskalującą, aby pozyskać lepsze cechy funkcji bazowych. Falki wielokrotne umożliwiają bowiem aproksymację wyższego rzędu z krótszą bazą w porównaniu z klasyczną funkcją skalującą. Ponadto możliwe jest skonstruowanie symetrycznych ortogonalnych falek wielokrotnych, co nie jest możliwe w przypadku klasycznych falek. W poprzednich latach pojawiło się kilka prostych konstrukcji baz falkowych opartych na hermitowskich splajnach kubicznych. W niniejszym artykule krótko przedstawiono takie konstrukcje, omawiane falki zastosowano do numerycznego rozwiązywania równań różniczkowych oraz porównano ich cechy.