APPLICATION OF RECONSTRUCTION OPERATORS IN THE DISCONTINUOUS GALERKIN METHOD

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Abstract

This paper gives an overview of the main ingredients needed to incorporate reconstruction operators, as known from higher order finite volume (FV) and spectral volume (SV) schemes, into the discontinuous Galerkin (DG) method. Such an operator constructs higher order approximations from the lower order DG scheme, increasing the order of convergence, while leading to a more efficient numerical scheme than the corresponding higher order DG scheme itself. We discuss theoretical, as well as implementational aspects and numerical experiments.

Keywords: Higher order reconstruction; discontinuous Galerkin; finite volumes.

Introduction

During the last decades, the finite volume (FV) method has gained the position of the standard method used by the engineering community for the discretization of equations governing conservation laws and convection-dominated problems. This method has many advantages, however one of its drawbacks is its low order accuracy. The standard approach to overcome this problem is the introduction of higher order reconstruction operators into the FV scheme. Although such an approach has not yet been theoretically justified, it gives excellent results in practice. However, one usually uses at most piecewise quadratic reconstructions, since higher orders are impractical and cumbersome from the implementational point of view.

A somewhat different approach to higher order schemes is the discontinuous Galerkin (DG) method, which combines concepts form the finite element and finite volume methods. Again, this method has many advantages, but one major drawback. While arbitrary orders of convergence can be achieved, the number of degrees of freedom needed grows very fast.

In the presented paper, we give an overview of a relatively new idea, originating in [2, 5], to combine the DG method with reconstruction operators to obtain a numerical scheme of very high orders of accuracy, which, we demonstrate, is computationally more efficient than the DG scheme itself. We introduce concepts needed to introduce such a scheme, discuss implementation and numerical results, as well as some theoretical considerations.

1 Problem Formulation and Notation

For simplicity, we shall be concerned with a scalar hyperbolic equation, although the same arguments basically hold for any time-dependent PDE. We treat a nonlinear nonstationary scalar hyperbolic equation in a bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz-continuous boundary $\partial\Omega$. We seek $u : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + \mathrm{div}\,\mathbf{f}(\mathrm{u}) = 0 \quad \text{in } \Omega \times (0,\mathrm{T}) \tag{1}$$

along with an appropriate initial and boundary condition. Here $\mathbf{f} = (f_1, \cdots, f_d)$ and $f_s, s = 1, \ldots, d$ are Lipschitz continuous fluxes in the direction $x_s, s = 1, \ldots, d$.

Let $\mathscr{T}_h$ be a partition (triangulation) of the closure $\overline{\Omega}$ into a finite number of closed simplices $K \in \mathscr{T}_h$. In general we do not require the standard conforming properties of $\mathscr{T}_h$ used in the finite element method (i.e. we admit the so-called hanging nodes). We shall use the following notation. By ∂K we denote the boundary of an element $K \in \mathscr{T}_h$ and set $h_K = \mathrm{diam}(K)$, $h = \max_{K \in \mathscr{T}_h} h_K$.

Let $K, K' \in \mathscr{T}_h$. We say that K and K' are *neighbours*, if they share a common *face* $\Gamma \subset \partial K$. By $\mathscr{F}_h$ we denote the system of all faces of all elements $K \in \mathscr{T}_h$. Further, we define the set of all interior and boundary faces, by $\mathscr{F}_h^I$ and $\mathscr{F}_h^B$, respectively.

For each $\Gamma \in \mathscr{F}_h$ we define a unit normal vector $\mathbf{n}_\Gamma$, such that for $\Gamma \in \mathscr{F}_h^B$ the normal $\mathbf{n}_\Gamma$ has the same orientation as the outer normal to $\partial\Omega$.

Over a triangulation $\mathscr{T}_h$ we define the *broken Sobolev spaces*

$$H^k(\Omega, \mathscr{T}_h) = \{v;\, v|_K \in H^k(K), \forall K \in \mathscr{T}_h\}.$$

For each face $\Gamma \in \mathscr{F}_h^I$ there exist two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathscr{T}_h$ such that $\Gamma \subset K_\Gamma^{(L)} \cap K_\Gamma^{(R)}$. We use the convention that $\mathbf{n}_\Gamma$ is the outer normal to $K_\Gamma^{(L)}$. For $v \in H^1(\Omega, \mathscr{T}_h)$ and $\Gamma \in \mathscr{F}_h^I$ we introduce the following notation:

$$v|_\Gamma^{(L)} = \text{ trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma, \quad v|_\Gamma^{(R)} = \text{ trace of } v|_{K_\Gamma^{(R)}} \text{ on } \Gamma, \quad [v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}.$$

On boundary edges we define $v|_\Gamma^{(R)} = [v]_\Gamma := v|_\Gamma^{(L)}$.

Let $n \geq 0$ be an integer. We define the space of discontinuous piecewise polynomial functions

$$S_h^n = \{v;\, v|_K \in P^n(K), \forall K \in \mathscr{T}_h\},$$

where $P^n(K)$ is the space of all polynomials on K of degree $\leq n$. Specifically,

- S_h^0: is the space of piecewise constant functions as known from the FV method,
- $S_h^n,\ n \geq 0$: the DG solution lies in this space of piecewise nth degree polynomials,
- $S_h^N,\ N > n$: the higher order reconstructed DG solution will lie in this space.

2 Discontinuous Galerkin

We multiply (1) by an arbitrary $\varphi_h^n \in S_h^n$, integrate over an element $K \in \mathscr{T}_h$ and apply Green's theorem. By summing over all $K \in \mathscr{T}_h$ and rearranging, we get

$$\frac{d}{dt} \int_\Omega u(t)\, \varphi_h^n \, dx + \sum_{\Gamma \in \mathscr{F}_h} \int_\Gamma \mathbf{f}(u) \cdot \mathbf{n}\, [\varphi_h^n]\, dS - \sum_{K \in \mathscr{T}_h} \int_K \mathbf{f}(u) \cdot \nabla \varphi_h^n \, dx = 0. \tag{2}$$

The boundary convective terms will be treated similarly as in the finite volume method, i.e. with the aid of a numerical flux $H(u, v, \mathbf{n})$:

$$\int_\Gamma \mathbf{f}(u) \cdot \mathbf{n}\, [\varphi_h^n]\, dS \approx \int_\Gamma H(u^{(L)}, u^{(R)}, \mathbf{n})\, [\varphi_h^n]\, dS. \tag{3}$$

We assume that H is *Lipschitz continuous, consistent* and *conservative*, cf. [3].

Finally, we define the *convective form* $b_h(\cdot, \cdot)$ defined for $v, \varphi \in H^1(\Omega, \mathscr{T}_h)$:

$$b_h(v, \varphi) = \sum_{\Gamma \in \mathscr{F}_h} \int_\Gamma H(v^{(L)}, v^{(R)}, \mathbf{n})\, [\varphi]\, dS - \sum_{K \in \mathscr{T}_h} \int_K \mathbf{f}(v) \cdot \nabla \varphi \, dx.$$

Definition 1 (Standard DG scheme) *We seek* $u_h : [0,T] \to S_h^n$ *such that*

$$\frac{d}{dt}\big(u_h(t), \varphi_h^n\big) + b_h\big(u_h(t), \varphi_h^n\big) = 0, \quad \forall \varphi_h^n \in S_h^n, \forall t \in (0,T). \tag{4}$$

We note that if we take $n = 0$, i.e. $u_h : (0,T) \to S_h^0$, then from the definition of b_h, we see that the DG scheme (4) is equivalent to the standard FV method.

3 Reconstructed Discontinuous Galerkin

For $v \in L^2(\Omega)$, we denote by $\Pi_h^n v$ the $L^2(\Omega)$-projection of v on S_h^n:

$$\Pi_h^n v \in S_h^n, \quad (\Pi_h^n v - v, \varphi_h^n) = 0, \qquad \forall\, \varphi_h^n \in S_h^n. \tag{5}$$

Obviously, if $K \in \mathscr{T}_h$, then the function $(\Pi_h^n v)|_K$ is the $L^2(K)$-projection of $v|_K$ on $P^n(K)$. The basis of the method lies in the observation that (2) can be viewed as an equation for the evolution of $\Pi_h^n u(t)$, where u is the exact solution of (1). In other words, due to (5), $\Pi_h^n u(t) \in S_h^n$ satisfies the following equation for all $\varphi_h^n \in S_h^n$:

$$\frac{d}{dt}\int_\Omega \Pi_h^n u(t)\, \varphi_h^n \, dx + \sum_{\Gamma \in \mathscr{F}_h} \int_\Gamma \mathbf{f}(u) \cdot \mathbf{n}\, [\varphi_h^n]\, dS - \sum_{K \in \mathscr{T}_h} \int_K \mathbf{f}(u) \cdot \nabla \varphi_h^n \, dx = 0. \tag{6}$$

Now, let $N > n$ be an integer. We assume that there exists a piecewise polynomial function $U_h^N(t) \in S_h^N$, which is an approximation of $u(t)$ of order $N+1$, i.e.

$$U_h^N(x,t) = u(x,t) + O(h^{N+1}), \quad \forall x \in \Omega, \forall t \in [0,T]. \tag{7}$$

This is possible, if u is sufficiently regular in space, e.g. $u(t) \in W^{N+1,\infty}(\Omega)$, cf.[1]. Now we incorporate the approximation $U_h^N(t)$ into (6): the exact solution u satisfies

$$\frac{d}{dt}\big(\Pi_h^n u(t), \varphi_h^n\big) + b_h\big(U_h^N(t), \varphi_h^n\big) = E(\varphi_h^n, t), \quad \forall \varphi_h^n \in S_h^n, \forall t \in (0,T), \tag{8}$$

where $E(\varphi_h^n, t)$ is an error term defined as

$$E(\varphi_h^n, t) = b_h\big(U_h^N(t), \varphi_h^n\big) - b_h\big(u(t), \varphi_h^n\big). \tag{9}$$

Lemma 1 *The following estimate holds for all* $t \in [0,T]$*:*

$$E(\varphi_h^n, t) = O(h^N) \|\varphi_h^n\|_{L^2(\Omega)}. \tag{10}$$

Proof: Due to the consistency and Lipschitz continuity of H, we have on $\Gamma \in \mathscr{F}_h$

$$\mathbf{f}(u) \cdot \mathbf{n} - H(U_h^{N,(L)}, U_h^{N,(R)}, \mathbf{n}) = H(u,u,\mathbf{n}) - H(U_h^{N,(L)}, U_h^{N,(R)}, \mathbf{n}) = O(h^{N+1}).$$

Furthermore, due to the Lipschitz-continuity of $\mathbf{f}$, we have on element $K \in \mathscr{T}_h$

$$\mathbf{f}(u) - \mathbf{f}(U_h^N) = O(h^{N+1}).$$

Estimate (10) follows from these results and the application of the *inverse* and *multiplicative trace inequalities*, cf. [3]. □

It remains to construct a sufficiently accurate approximation $U_h^N(t) \in S_h^N$ to $u(t)$, such that (7) is satisfied. This leads to the following problem.

Definition 2 (Reconstruction problem) *Let $v : \Omega \to \mathbb{R}$ be sufficiently regular. Given $\Pi_h^n v \in S_h^n$, find $v_h^N \in S_h^N$ such that $v - v_h^N = O(h^{N+1})$ in Ω. We define the corresponding reconstruction operator $R : S_h^n \to S_h^N$ by $R\Pi_h^n v := v_h^N$.*

By setting $U_h^N(t) := R\Pi_h^n u(t)$ in (8), we obtain the following equation for the $L^2(\Omega)$-projections of the exact solution u onto the space S_h^n:

$$\frac{d}{dt}\left(\Pi_h^n u(t), \varphi_h^n\right) + b_h\left(R\Pi_h^n u(t), \varphi_h^n\right) = O(h^N)\|\varphi_h^n\|_{L^2(\Omega)}, \quad \forall \varphi_h^n \in S_h^n. \tag{11}$$

By neglecting the right-hand side and approximating $u_h^n(t) \approx \Pi_h^n u(t)$, we arrive at the following definition of the *reconstructed discontinuous Galerkin* (RDG) scheme.

Definition 3 (Reconstructed DG scheme) *We seek $u_h^n : [0,T] \to S_h^n$ such that*

$$\frac{d}{dt}\left(u_h^n(t), \varphi_h^n\right) + b_h\left(Ru_h^n(t), \varphi_h^n\right) = 0, \quad \forall \varphi_h^n \in S_h^n, \forall t \in (0,T). \tag{12}$$

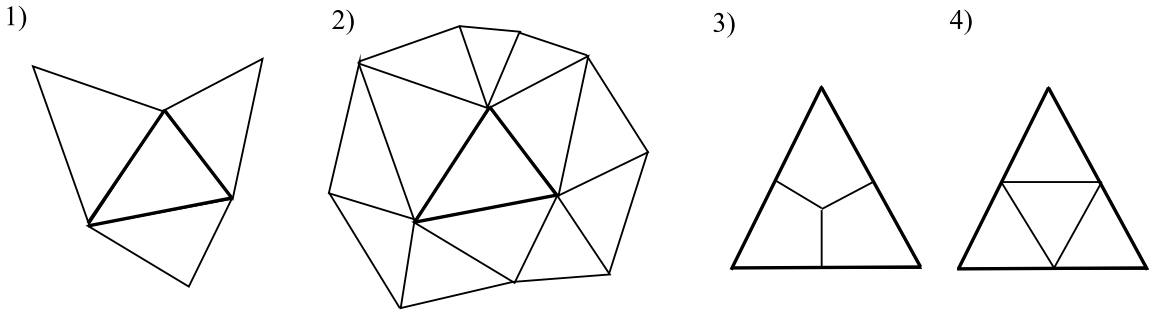
There are several points worth mentioning.

- Equation (11) indicates that the RDG scheme is formally Nth order in space.
- The derivation of the RDG scheme follows the methodology of higher order FV and SV schemes, cf. [7]. The basis of these schemes is an equation for the evolution of averages of the exact solution on individual elements (i.e. an equation for $\Pi_h^0 u(t)$). Equation (11) is a direct generalization for the case of higher order $L^2(\Omega)$-projections $\Pi_h^n u(t)$, $n \geq 0$.
- Both $u_h^n(t)$ and φ_h^n lie in S_h^n. Only $Ru_h^n(t)$, lies in the higher dimensional space S_h^N. Despite this fact, equation (11) indicates that we may expect $u - Ru_h^n = O(h^{N+1})$, although $u - u_h^n = O(h^{n+1})$.
- Numerical quadrature must be employed to evaluate surface and volume integrals in (12). Since test functions are in S_h^n, as compared to S_h^N in the corresponding Nth order standard DG scheme, we may use lower order (i.e. more efficient) quadrature formulae as compared to the standard DG method.
- In practice, we must also discretize (12) with respect to time. As in the case of higher order FV methods, we use an explicit time stepping method. The upper limit on stable time steps, given by a CFL-like condition, is more restrictive with growing N. However, in the RDG scheme, stability properties are essentially inherited from the lower order scheme, therefore a larger time step is possible as compared to the corresponding Nth order standard DG scheme.

3.1 Explicit Time Discretization

For simplicity, we formulate the forward Euler method, which is only the first order accurate, however in Section 5, higher order Adams-Bashforth methods are used.

Let us construct a partition $0 = t_0 < t_1 < t_2 \ldots$ of the time interval $[0,T]$ and define the time step $\tau_k = t_{k+1} - t_k$. We use the approximation $u_h^{n,k} \approx u_h^n(t_k)$, where $u_h^{n,k} \in S_h^n$. The forward Euler scheme is given by:



Source: Own

Fig. 1. *1) FV stencil for linear reconstruction, 2) FV stencil for quadratic reconstruction, 3) Control volumes in a spectral volume for linear reconstruction, 4) Analogy to the SV approach for DG methods - partition of triangle into control volumes, e.g. cubic reconstruction from linear data*

Definition 4 (Explicit RDG scheme) *We seek* $u_h^{n,k} \in S_h^n,\ k = 0,1,\ldots$ *such that*

$$\left(\frac{u_h^{n,k+1} - u_h^{n,k}}{\tau_k}, \varphi_h^n\right) + b_h\left(Ru_h^{n,k}, \varphi_h^n\right) = 0, \quad \forall \varphi_h^n \in S_h^n,\ k = 0,1,\ldots, \tag{13}$$

where $u_h^{n,0} = u_{h,0}$ *is an* S_h^n *approximation of the initial condition* u^0.

3.2 Construction of the Reconstruction Operator

In analogy to the construction of reconstruction operators in higher order FV schemes, we propose two approaches.

3.2.1 ”Standard” Approach

In the *standard approach*, a stencil (a group of neighboring elements and the element under consideration) is used to build an Nth degree polynomial approximation to u on the element under consideration ([4, 6]). In the FV method, the von Neumann neighborhood of an element is used as a stencil to obtain a piecewise linear reconstruction, cf. Figure 1, 1). However, for higher order reconstructions, the size of the stencil increases dramatically, cf. Figure 1, 2), causing higher degrees than quadratic to be very time consuming. In the case of the RDG scheme, we need not increase the stencil size to obtain higher order accuracy, it suffices to take the von Neumann neighborhood and increase the order of the underlying DG scheme.

In analogy to the FV method, the reconstruction operator R is constructed on each stencil independently and satisfies that $R\Pi_h^n$ is in some sense *polynomial preserving*. Specifically, for each element K and its corresponding stencil S, we require that for all $p \in P^N(S)$

$$\left(\left(R\Pi_h^n\right)\big|_S p\right)\Big|_K = p\big|_K. \tag{14}$$

This requirement allows us to study approximation properties of R using the Bramble–Hilbert technique as in the standard finite element method, [1]. The disadvantage of this approach is that for unstructured meshes, the coefficients of the reconstruction operator must be stored for each individual stencil.

In the FV method, different conditions on R than (14) are often used, e.g. continuous or discrete least squares. Special care must be taken in the vicinity of steep gradients and discontinuities, where the Gibbs phenomenon may occur. In this case different strategies are employed, e.g. limiting, ENO and WENO schemes, TVD etc. The generalization of these concepts to the RDG method is left for future work.

3.2.2 Spectral Volume Approach

In the *spectral volume approach*, we start with a partition of Ω into so-called *spectral volumes* S, for example triangles in 2D. The triangulation $\mathscr{T}_h$ is formed by subdividing each spectral volume S into sub-cells K, called *control volumes*, cf. [7]. In the FV method, the order of accuracy of the reconstruction determines the number of control volumes to be generated in each spectral volume. For example, for a linear reconstruction on a triangle, the triangle is divided into three control volumes, Figure 1, 3). Again, in the RDG scheme, we may use only the smallest available partition into control volumes, and increase the accuracy by increasing the order of the underlying scheme, cf. Figure 1, 4).

The reconstruction operator is constructed on each spectral volume independently such that it is in some sense polynomial preserving, i.e. for each spectral volume S, we require that for all $p \in P^N(S)$

$$\left(R\Pi_h^n\right)\big|_S p = p. \tag{15}$$

The advantage of this approach is that all spectral volumes are affine equivalent, we construct the reconstruction operator R only on one reference spectral volume.

4 Relation between RDG and Standard DG

The only difference between the DG scheme (4) and RDG scheme (12) is the presence of the reconstruction operator R in the first variable of $b_h(\cdot,\cdot)$. While the error analysis of (4) is well understood (at least for convection-diffusion problems [3]), the analysis of (12) or (13) poses a new challenge. The problem lies in the fact that we cannot test (12) with $\varphi_h^n := Ru_h^{n,k}$ or something similar, since $Ru_h^{n,k} \notin S_h^n$. Therefore, we need to establish a relation between (12) and the Nth order DG method, instead of only the nth order DG method.

Definition 5 (Auxiliary problem) *We seek $\tilde{u}_h^{N,k} \in S_h^N$ such that*

$$\left(\frac{\tilde{u}_h^{N,k+1} - \tilde{u}_h^{N,k}}{\tau_k}, \varphi_h^N\right) + b_h\left(R\Pi_h^n \tilde{u}_h^{N,k}, \varphi_h^N\right) = 0, \quad \forall \varphi_h^N \in S_h^N,\ k = 0,1,\ldots, \tag{16}$$

where $\tilde{u}_h^{N,0}$ is an S_h^N approximation of the initial condition u^0.

Lemma 2 *Let $u_h^{n,0} = \Pi_h^n \tilde{u}_h^{N,0}$. Then $u_h^{n,k} \in S_h^n$, the solution of (13) and the solution $\tilde{u}_h^{N,k} \in S_h^N$ of (16) satisfy*

$$u_h^{n,k} = \Pi_h^n \tilde{u}_h^{N,k}, \quad \forall k = 0,1,\cdots. \tag{17}$$

Proof: We prove (17) by induction:

$k = 1$: Since $u_h^{n,0} = \Pi_h^n \tilde{u}_h^{N,0}$, we have for all $\varphi_h^n \in S_h^n$

$$\begin{aligned}(\Pi_h^n \tilde{u}_h^{N,1}, \varphi_h^n) &= (\tilde{u}_h^{N,1}, \varphi_h^n) = (\tilde{u}_h^{N,0}, \varphi_h^n) - \tau_k b_h\left(R\Pi_h^n \tilde{u}_h^{N,0}, \varphi_h^n\right) \\ &= (u_h^{n,0}, \varphi_h^n) - \tau_k b_h\left(Ru_h^{n,0}, \varphi_h^n\right) = (u_h^{n,1}, \varphi_h^n),\end{aligned}$$

hence $(\Pi_h^n \tilde{u}_h^{N,1} - u_h^{n,1}, \varphi_h^n) = 0$ for all $\varphi_h^n \in S_h^n$. Therefore $\Pi_h^n \tilde{u}_h^{N,1} = u_h^{n,1}$.

$k > 1$: Assume (17) holds for some $k > 1$. Then for all $\varphi_h^n \in S_h^n$

$$\begin{aligned}\left(\Pi_h^n \tilde{u}_h^{N,k+1}, \varphi_h^n\right) &= \left(\tilde{u}_h^{N,k+1}, \varphi_h^n\right) = \left(\tilde{u}_h^{N,k}, \varphi_h^n\right) - \tau_k b_h\left(R\Pi_h^n \tilde{u}_h^{N,k}, \varphi_h^n\right) \\ &= \left(u_h^{n,k}, \varphi_h^n\right) - \tau_k b_h\left(R u_h^{n,k}, \varphi_h^n\right) = \left(u_h^{n,k+1}, \varphi_h^n\right),\end{aligned}$$

therefore $\Pi_h^n \tilde{u}_h^{N,k+1} = u_h^{n,k+1}$. This completes the induction step $k \to k+1$. □

As a corollary, error estimates for the auxiliary problem imply error estimates for the RDG scheme (12). Problem (16) is basically the standard Nth order DG scheme with the operator $R\Pi_h^n$ in the first variable of $b_h(\cdot,\cdot)$. Therefore, sufficient knowledge of the properties of $R\Pi_h^n$ (which is polynomial preserving) and standard DG method error estimates would imply the estimates for the RDG scheme.

5 Numerical Experiments

We present numerical experiments for the periodic advection of a 1D sine wave on uniform meshes. We use the P^1 RDG scheme with P^5 reconstruction and the P^2 RDG scheme with P^8 reconstruction. The reconstruction operators described in Section 3.2.1 are used. Experimental orders of accuracy α in various norms on meshes with N elements are given in Tables 1 and 2. Here $e_h = u - Ru_h^n$ at t corresponding to ten periods. The increase in accuracy due to reconstruction is clearly visible.

Tab. 1. *1D advection of sine wave, P^1 RDG scheme with P^5 reconstruction*

N	$\|\|e_h\|\|_{L^\infty(\Omega)}$	α	$\|\|e_h\|\|_{L^2(\Omega)}$	α	$\|e_h\|_{H^1(\Omega,\mathscr{T}_h)}$	α
4	5.82E-03	–	3.49E-03	–	3.65E-02	–
8	7.53E-05	6.27	4.43E-05	6,30	1.06E-03	5,11
16	9.07E-07	6.38	5.95E-07	6,22	3.58E-05	4,89
32	1.82E-08	5.64	8.70E-09	6,10	1.16E-06	4,95
64	3.41E-10	5.74	1.33E-10	6,03	3.67E-08	4,98

Source: Own

Tab. 2. *1D advection of sine wave, P^2 RDG scheme with P^8 reconstruction*

N	$\|\|e_h\|\|_{L^\infty(\Omega)}$	α	$\|\|e_h\|\|_{L^2(\Omega)}$	α	$\|e_h\|_{H^1(\Omega,\mathscr{T}_h)}$	α
4	2.90E-03	–	1.85E-03	–	1.63E-02	–
8	7.75E-06	8.55	3.56E-06	9.02	1.03E-04	7.30
16	2.10E-08	8.53	6.64E-09	9.07	4.34E-07	7.89
32	7.21E-11	8.18	4.02E-11	7.37	1.76E-09	7.94

Source: Own

Conclusion

We have presented a possible generalization of higher order reconstruction operators as used in the FV method to the DG method. Two constructions of the reconstruction operator R are presented, the first analogous to the standard FV case (already treated in [2]) and the construction analogous to the SV method. The resulting scheme has many advantages over standard DG, FV and SV schemes:

- To increase the order of the scheme, the reconstruction stencil need not be enlarged, we may simply increase the order of the underlying DG scheme.
- Test functions are from the lower order space, hence more efficient quadratures may be used than in the corresponding higher order DG scheme.
- Since the RDG scheme is basically a lower order DG scheme with higher order reconstruction, the CFL condition is less restrictive than for the corresponding higher order DG scheme.

Acknowledgments

This work was supported by the Grant No. P201/11/P414 of the Czech Science Foundation. The author is a (junior) researcher (or member) of the University center for mathematical modelling, applied analysis and computational mathematics (Math MAC).

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RNDr. Václav Kučera, Ph.D.

Aplikace rekonstrukčních operátorů v nespojité Galerkinově metodě

Tento článek představuje přehled základních konceptů nutných k zapojení rekonstrukčních operátorů, známých z metody konečných objemů (FV) a spektrálních objemů (SV) vyššího řádu, do nespojité Galerkinovy (DG) metody. Takovýto operátor konstruuje aproximace vyšších řádů z DG metody nižšího řádu, čímž zvyšuje řád konvergence, zároveň vede k efektivnějšímu numerickému schématu než příslušné DG schéma vyššího řádu samo o sobě. Probereme teoretické i implementační aspekty a numerické experimenty.

Die Anwendung von Rekonstruktionsoperatoren in der kontinuierlichen Galerkin-Methode

Dieser Artikel bietet einen Überblick über die grundlegenden, zu Einbeziehung der Rekonstruktionsoperatoren notwendigen Konzepte, die aus der Methode des finiten Inhalts (FV) und der spektralen Inhalte (SV) höherer Ordnung bekannt sind, in die nicht kontinuierliche Galerkin-Methode (DG). Ein solcher Operator konstruiert eine Annäherung hherer Ordnungen aus der DG-Methode niederer Ordnung, wodurch er die Ordnung der Konvergenz erhöht. Gleichzeitig führt er zu einem effektiveren numerischen Schema als das zugehörige DG-Schema höherer Ordnung selbst. Wir behandeln die theoretischen und implementären Aspekte und die numerischen Experimente.

Zastosowanie operatorów rekonstrukcyjnych w nieciągłej metodzie Galerkina

W niniejszym artykule przedstawiono podstawowe koncepcje niezbędne do włączenia operatorów rekonstrukcyjnych, znanych z metody objętości skończonych (FV) oraz objętości spektralnych (SV) wyższego rzędu, do nieciągłej metody Garlekina (DG). Taki operator konstruuje aproksymacje wyższego rzędu z metody DG niższego rzędu, co zwiększa poziom konwergencji i jednocześnie prowadzi do bardziej efektywnego schematu numerycznego w porównaniu ze schematem DG wyższego rzędu. Omówiono aspekty teoretyczne i implementacyjne oraz eksperymenty numeryczne.