NUMERICAL EVALUATION OF RHEOLOGICAL EXPERIMENT

Petr Salač
***Ivo Matoušek**

Technical University of Liberec
Faculty of Science, Humanities and Education
Studentská 2, 461 17, Liberec, Czech Republic
petr.salac@tul.cz

*Technical University of Liberec
Faculty of Mechanical Engineering
Studentská 2, 461 17, Liberec, Czech Republic
ivo.matousek@tul.cz

Abstract

A viscoelastic simply supported rotationally symmetric body, fixed on a base, is considered. The body is loaded by a flat plunger, which moves in the direction of the z axis by a constant velocity v. In this work the reaction force is computed. This allows us to compare numerical results with data from rheological experiment (see [6], [7]). The variational formulation of the problem is derived and transformed to cylindrical coordinates. Some results of numerical calculations are presented.

Keywords: Viscoelasticity; axisymmetric hyperbolic problems; dimensional reduction.

Introduction

Mathematical modeling of technological processes has been already regarded as a powerful tool for an optimization of technological processes also in glass industry. The fundamental issue of virtual modelling of silica glass forming is an accuracy of numerical outputs. Critical factor is not only definition of boundary conditions, mainly thermal ones, but also specification of material properties. Number of methods for an identification of rheological properties exists [1]. The disadvantage of the most of published models is independent description of the both stages - the stage with the dominant influence of an elastic component of deformation and that one with a dominant viscous flow [8]. One of the most effective methods is isothermal compression method which is based on the evaluation of the force response on compression loading of cylindrical samples [4], [6], [7].

The advantage of this method is its relatively simplicity and possibility to evaluate both elastic and viscous properties of glass melt simultaneously during one experiment. However the critical issue of this method is an accuracy of evaluation of experimental outputs. Several methods were suggested.

In this contribution we introduce the variational formulation of the rheological experiment model, which includes viscoelastic deformations. This problem will be used as a state problem in formulations of various identification problems for various model parameters, that are planed as next step of our research. Nevertheless recent numerical results are roughly in conformity with results of experiments (see [6]).

1 Formulation of the Problem

Forming of glass is quite complicated process which contains both elastic and plastic responses to strain from stress. This is the main reason of using a viscoelastic model to describe relationship between stress and strain.

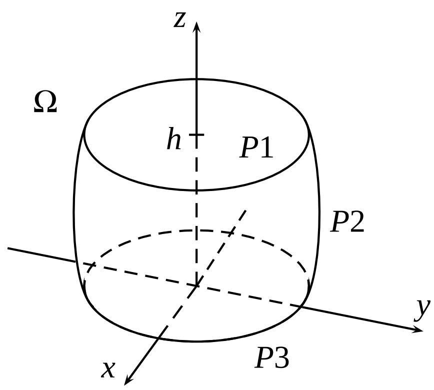

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Fig. 1. *Scheme of glass sample*

We consider a viscoelastic isotropic homogeneous cylindrical body Ω symmetrical according to z axis with bases formed by two parallel circles with radii R, and altitude h. We consider the body which is fixed on both bases and which is free on its surrounding surface. We denote P_1 upper, resp. P_3 bottom, base and P_2 surrounding surface of the body. The body is deformed by flat plunger moving by a constant velocity v in the direction of the z axis placed on the upper base. To represent changes of the shape of the body we define a deformation tensor by the formula

$$\varepsilon_{ij}(\boldsymbol{x},t) = \frac{1}{2}\left(\frac{\partial u_i(\boldsymbol{x},t)}{\partial x_j} + \frac{\partial u_j(\boldsymbol{x},t)}{\partial x_i}\right) . \tag{1}$$

The stress is represented by symmetrical stress tensor σ. We consider stress strain relation given by viscoelastic generalized Hook's law in the form

$$\sigma_{ij}(\boldsymbol{x},t) = \delta_{ij}\int_{-\infty}^{t} \lambda(t-\tau)\frac{\partial \varepsilon_{kk}(\boldsymbol{x},\tau)}{\partial \tau}\,d\tau + 2\int_{-\infty}^{t} \mu(t-\tau)\frac{\partial \varepsilon_{ij}(\boldsymbol{x},\tau)}{\partial \tau}\,d\tau , \tag{2}$$

where $\lambda(t)$ and $\mu(t)$ denote relaxation functions describing glass properties in pressing, resp. in shear and δ_{ij} is the Kroneker symbol.
The balance of a linear momentum for the dynamic problem has the form

$$\frac{\partial \sigma_{ij}(\boldsymbol{x},t)}{\partial x_j} + \mathscr{F}_i(\boldsymbol{x},t) = \rho\frac{\partial^2 u_i(\boldsymbol{x},t)}{\partial t^2} \quad i = 1,2,3 , \tag{3}$$

where $\mathscr{F}(\boldsymbol{x},t)$ represents the body forces and ρ the density of glass.
We use the time discretization method:
Let the time interval $[0,T]$ be divided into the subintervals $[t_{k-1},t_k]$, for $k = 1,2,\ldots,p$, then (3) has at each time level the form

$$\frac{\partial \sigma_{ij}^k(\boldsymbol{x})}{\partial x_j} + F_i^k(\boldsymbol{x}) = \rho\frac{z_i^k(\boldsymbol{x}) - 2z_i^{k-1}(\boldsymbol{x}) + z_i^{k-2}(\boldsymbol{x})}{h^2} \qquad i = 1,2,3, \quad k = 2,\ldots,p, \tag{4}$$

where $z_i^k(\boldsymbol{x}) = u(\boldsymbol{x},t_k)$, $\sigma_{ij}^k(\boldsymbol{x}) = \sigma_{ij}(\boldsymbol{x},t_k)$, $F_i^k(\boldsymbol{x}) = \mathscr{F}_i(\boldsymbol{x},t_k)$ and $h = \frac{T}{p}$.
According to the fact that the body, support and load have rotational symmetry we transform

the problem to cylindrical coordinates and apply dimensional reduction of an angle to get two-dimensional problem.
We are going to solve the problem in the region $D^k(\alpha)$ dependent on time the level t_k bounded by the axis r (part Γ_3), the axis z (part Γ_4), the straight line $z = h - tv$ (part Γ_1) and the free boundary described by the function $\alpha(z)$ of variable z (part Γ_2).

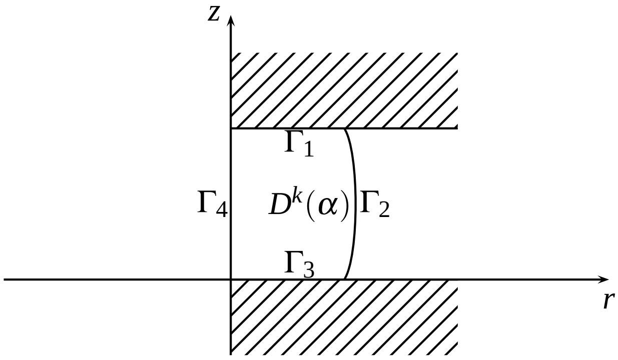

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Fig. 2. *Domain after the dimensional reduction*

According to the fact that the problem has rotational symmetry we assume that a displacement vector component in the direction ϑ is zero $(z^k_\vartheta(\boldsymbol{x}) = 0)$, similarly $\frac{\partial z^k_r}{\partial \vartheta} = 0$, and $\frac{\partial z^k_z}{\partial \vartheta} = 0$. We denote the physical components of the displacement vector by two functions, e.g.

$$\begin{aligned} z^k_r(\boldsymbol{x}) &= u^k, \\ z^k_z(\boldsymbol{x}) &= w^k, \\ (z^k_\vartheta(\boldsymbol{x}) &= 0). \end{aligned}$$

The relationship between the displacement vector and the strain tensor is in the form

$$\varepsilon^k_{rr} = \frac{\partial u^k}{\partial r}, \tag{5}$$

$$\varepsilon^k_{zz} = \frac{\partial w^k}{\partial z}, \tag{6}$$

$$\varepsilon^k_{\vartheta\vartheta} = \frac{u^k}{r}, \tag{7}$$

$$\varepsilon^k_{rz} = \frac{1}{2}\left(\frac{\partial u^k}{\partial z} + \frac{\partial w^k}{\partial r}\right), \tag{8}$$

$$(\varepsilon^k_{r\vartheta} = 0, \ \ \varepsilon^k_{z\vartheta} = 0). \tag{9}$$

The components of the stress tensor $\boldsymbol{\sigma}$ have the form

$$\sigma^k_{rr} = \frac{1}{h}\int_{-\infty}^{t} \lambda(t-\tau)(e^k(\boldsymbol{x}) - e^{k-1}(\boldsymbol{x})) + 2\mu(t-\tau)(\varepsilon^k_{rr}(\boldsymbol{x}) - \varepsilon^{k-1}_{rr}(\boldsymbol{x}))\,d\tau, \tag{10}$$

$$\sigma^k_{zz} = \frac{1}{h}\int_{-\infty}^{t} \lambda(t-\tau)(e^k(\boldsymbol{x}) - e^{k-1}(\boldsymbol{x})) + 2\mu(t-\tau)(\varepsilon^k_{zz}(\boldsymbol{x}) - \varepsilon^{k-1}_{zz}(\boldsymbol{x}))\,d\tau, \tag{11}$$

$$\sigma^k_{\vartheta\vartheta} = \frac{1}{h}\int_{-\infty}^{t} \lambda(t-\tau)(e^k(\boldsymbol{x}) - e^{k-1}(\boldsymbol{x})) + 2\mu(t-\tau)(\varepsilon^k_{\vartheta\vartheta}(\boldsymbol{x}) - \varepsilon^{k-1}_{\vartheta\vartheta}(\boldsymbol{x}))\,d\tau, \tag{12}$$

$$\sigma_{rz}^k = \frac{2}{h} \int_{-\infty}^{t} \mu(t-\tau)(\varepsilon_{rz}^k(\boldsymbol{x}) - \varepsilon_{rz}^{k-1}(\boldsymbol{x}))\, d\tau \,, \tag{13}$$

$$(\sigma_{r\vartheta}^k = 0, \quad \sigma_{z\vartheta}^k = 0) \,, \tag{14}$$

where

$$e = \varepsilon_{rr} + \varepsilon_{zz} + \varepsilon_{\vartheta\vartheta} \,. \tag{15}$$

The bilinear form representing mechanical work of inner forces has the form

$$\begin{aligned} A(\boldsymbol{u},\boldsymbol{\varphi}) \;=\; & -\frac{1}{h}\int_{D^k(\alpha)}\int_{-\infty}^{t} \Big((\lambda(t-\tau) + 2\mu(t-\tau)) \Big[\frac{\partial u^k(\boldsymbol{x})}{\partial r}\frac{\partial \varphi_1(\boldsymbol{x})}{\partial r} r + \\ & + \frac{\partial w^k(\boldsymbol{x})}{\partial z}\frac{\partial \varphi_2(\boldsymbol{x})}{\partial z} r \;+\; u^k(\boldsymbol{x})\varphi_1(\boldsymbol{x})\frac{1}{r} \Big] \;+ \\ & + \mu(t-\tau)\; \Big[\frac{\partial w^k(\boldsymbol{x})}{\partial r}\frac{\partial \varphi_2(\boldsymbol{x})}{\partial r} r \;+\; \frac{\partial w^k(\boldsymbol{x})}{\partial r}\frac{\partial \varphi_1(\boldsymbol{x})}{\partial z} r \;+ \\ & + \frac{\partial u^k(\boldsymbol{x})}{\partial z}\frac{\partial \varphi_2(\boldsymbol{x})}{\partial r} r \;+\; \frac{\partial u^k(\boldsymbol{x})}{\partial z}\frac{\partial \varphi_1(\boldsymbol{x})}{\partial z} r \Big] \Big)\, d\tau\, d\boldsymbol{x} \;+ \\ & + \frac{\rho}{h^2}\int_{D^k(\alpha)} \Big(u^k(\boldsymbol{x})\varphi_1(\boldsymbol{x}) r \;+ w^k(\boldsymbol{x})\varphi_2(\boldsymbol{x}) r \Big)\, d\boldsymbol{x} \,. \end{aligned} \tag{16}$$

Linear functional representing mechanical work of outward forces has the form

$$\begin{aligned} \langle \boldsymbol{F}(t),\boldsymbol{\varphi}\rangle \;=\; & \int_{D^k(\alpha)} \Big([\, F_1^k(\boldsymbol{x})\varphi_1(\boldsymbol{x}) r + F_2^k(\boldsymbol{x})\varphi_2(\boldsymbol{x}) r \,] - \\ & -\frac{1}{h}\int_{-\infty}^{t} \Big((\lambda(t-\tau) + 2\mu(t-\tau)) \Big[\frac{\partial u^{k-1}(\boldsymbol{x})}{\partial r}\frac{\partial \varphi_1(\boldsymbol{x})}{\partial r} r + \\ & + \frac{\partial w^{k-1}(\boldsymbol{x})}{\partial z}\frac{\partial \varphi_2(\boldsymbol{x})}{\partial z} r \;+\; u^{k-1}(\boldsymbol{x})\varphi_1(\boldsymbol{x})\frac{1}{r} \Big] \;+ \\ & + \mu(t-\tau)\; \Big[\frac{\partial w^{k-1}(\boldsymbol{x})}{\partial r}\frac{\partial \varphi_2(\boldsymbol{x})}{\partial r} r \;+\; \frac{\partial w^{k-1}(\boldsymbol{x})}{\partial r}\frac{\partial \varphi_1(\boldsymbol{x})}{\partial z} r \;+ \\ & + \frac{\partial u^{k-1}(\boldsymbol{x})}{\partial z}\frac{\partial \varphi_2(\boldsymbol{x})}{\partial r} r \;+\; \frac{\partial u^{k-1}(\boldsymbol{x})}{\partial z}\frac{\partial \varphi_1(\boldsymbol{x})}{\partial z} r \Big] \Big)\, d\tau \;- \\ & - \frac{\rho}{h^2}\Big[\Big(u^{k-2}(\boldsymbol{x})\varphi_1(\boldsymbol{x}) r \;+ w^{k-2}(\boldsymbol{x})\varphi_2(\boldsymbol{x}) r \Big) - \\ & -2\,\Big(u^{k-1}(\boldsymbol{x})\varphi_1(\boldsymbol{x}) r \;+ w^{k-1}(\boldsymbol{x})\varphi_2(\boldsymbol{x}) r \Big) \Big] \Big)\, d\boldsymbol{x} \,. \end{aligned} \tag{17}$$

Boundary conditions:

The flat plunger moves in the direction of the z axis by the constant velocity v acting on part of boundary Γ_1, i.e.

$$\left.\begin{aligned} u &= 0 \\ w &= v(t - t_k) \end{aligned}\right\} \;\text{ on }\quad \Gamma_1 \,. \tag{18}$$

The part of boundary Γ_2 represents the so called free boundary which is deformed by the influence of the inner forces and is not able to catch any force (tangent or normal)

$$\left.\begin{aligned} \sigma_{rr} &= 0 \\ \sigma_{zz} &= 0 \end{aligned}\right\} \;\text{ on }\quad \Gamma_2 \,. \tag{19}$$

The part of boundary Γ_3 is fixed, i.e.

$$\left.\begin{array}{l} u = 0 \\ w = 0 \end{array}\right\} \quad \text{on} \quad \Gamma_3 \, . \tag{20}$$

The part of boundary Γ_4 is formed by the axis of symmetry and has properties of contact with solid support (without friction), i.e.

$$\left.\begin{array}{l} u = 0 \\ \frac{\partial w}{\partial r} = 0 \end{array}\right\} \quad \text{on} \quad \Gamma_4 \, . \tag{21}$$

We define the space $W^{1,2,r}(D^k(\alpha))$ with the norm

$$\|u\|_{1,2,r} \;=\; \left(\int_{D^k(\alpha)} \left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + u^2\right] r \, d\boldsymbol{x}\right)^{\frac{1}{2}} \, . \tag{22}$$

We define the space of functions with the finite energy as the weighted Sobolev space $\mathscr{H}(D^k(\alpha))$

$$\mathscr{H}(D^k(\alpha)) = \{\hat{\boldsymbol{u}} \equiv (u, w) \in W^{1,2,r}(D^k(\alpha)) \times W^{1,2,r}(D^k(\alpha))\} \, . \tag{23}$$

We denote

$$\mathscr{V}_1 = \{u \in C^\infty(D^k(\alpha)) \mid \operatorname{supp} u \cap \Gamma_4 = \emptyset, \; u = 0 \;\text{ on }\; \Gamma_1 \cup \Gamma_3\} \, . \tag{24}$$

Let V_1 be the closure of the set $\mathscr{V}_1$ in the space $W^{1,2,r}(D^k(\alpha))$.
Further we denote

$$\mathscr{V}_2 = \{u \in C^\infty(D^k(\alpha)) \mid u = 0 \;\text{ on }\; \Gamma_1 \cup \Gamma_3\} \, . \tag{25}$$

Let V_2 be the closure of the set $\mathscr{V}_2$ in the space $W^{1,2,r}(D^k(\alpha))$.
We denote by

$$\boldsymbol{H} = \{\hat{\boldsymbol{u}} \equiv (u, w) \in V_1 \times V_2\} \tag{26}$$

the space of test functions (i.e. such functions with finite energy which satisfy stable boundary conditions).
We use the principle of virtual displacement to get a variational formulation of the problem:
Let $\hat{u}_0 \in \mathscr{H}(D^k(\alpha))$ be given, which specifies the displacement on the boundary Γ_1 by its traces. We are looking for $\hat{u} \in \mathscr{H}(D^k(\alpha))$ such that

$$\hat{\boldsymbol{u}} - \hat{\boldsymbol{u}}_0 \;\in\; \boldsymbol{H} \, , \tag{27}$$

$$A(\hat{\boldsymbol{u}}, \hat{\boldsymbol{\varphi}}) \;=\; \langle \boldsymbol{F}(t), \hat{\boldsymbol{\varphi}} \rangle \qquad \forall \hat{\boldsymbol{\varphi}} \in \boldsymbol{H}, \;\; k = 2, 3, \ldots, p \, . \tag{28}$$

Theorem. The problem (27) - (28) has the unique solution.

Proof: The proof based on the Lax-Milgram theorem is too long and technical to be published.

2 Numerical Experiment

The numerical model, describing the course of experimental measurements of rheological properties of melt glass, was created.

The principle of experiment is based on the evaluation of the force (viscoelastic) response of isothermal cylindrical molten glass sample, which is compressed at a constant velocity. Numerical simulation was realized in the commercial FEM (Finite Elements Method) code MSC MARC.

The initial sample sizes were 20,3 mm (diameter) - 18,45 mm (height), the velocities of compression were taken from the range 0,5 - 40 mm/s. The Maxwell model was used for description of material behavior of FLOAT melted glass.
Viscosity of the shaped glass was defined according to the experiment, i.e. $\eta = 10^{7,52}$ [Pa.s]. The modulus of elasticity was selected from the range $E_1 = 2,5.10^8 \ \ 2,5.10^9$ [Pa], molten glass was assumed to be incompressible substance, i.e. The Poisson constant $\nu = 0,5$.

Sticking conditions were presumed between glass and metal punch contact surfaces.

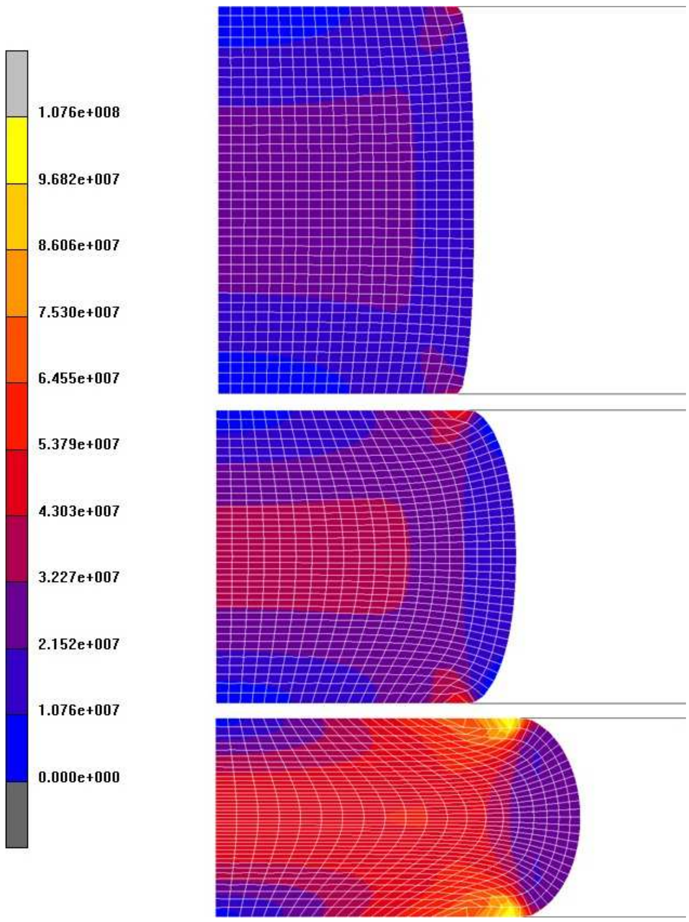

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Fig. 3. *Distribution of the stress fields in the form of equivalent Cauchy stress for compression 2, 6 and 10 mm*

The course of distribution of the stress fields in the form of the equivalent Cauchy stress for 3 different stages (compression 2, 6 and 10 mm) are presented in Fig. 3 (for velocity $v = 4$ [mm/s]).

The courses of the force response for different elastic moduli are shown in Fig. 4. From the figure it results that the elastic modulus influences only the first stage of the experiment, second one is only controlled by viscous flow.

Conclusion

In the contribution the model for evaluation of description of viscoelastic force response to compression loading was suggested. Integration of the viscoelastic model of the Maxwell type

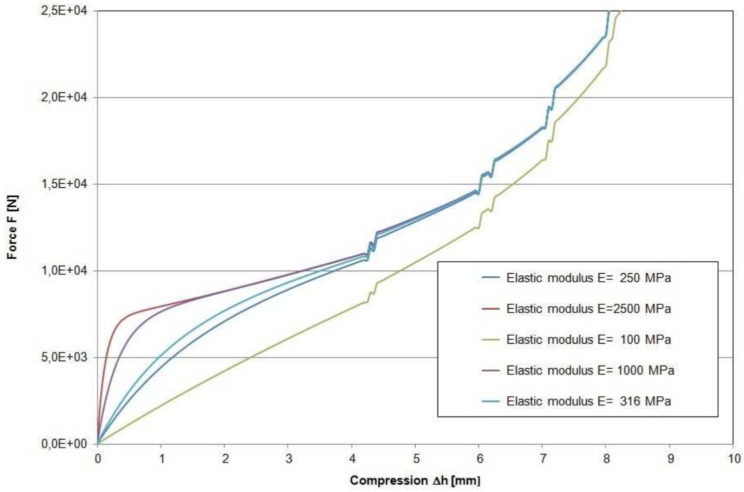

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Fig. 4. *Course of computed load forces*

to the mathematical model allowed fair description of the force response according to character of the realized experiments [6]. The shape course of deformed glass sample in the experiment was visibly similar to the computed one. Development of the measured load force showed the similar tendency but values became more different from the computed ones during the experiment.

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RNDr. Petr Salač, CSc., Ing. Ivo Matoušek, Ph.D.

NUMERICKÉ HODNOCENÍ REOLOGICKÉHO EXPERIMENTU

Uvažujeme viskoelastické, prostě podepřené, rotačně symetrické těleso pevně spojené s podkladem. Těleso je zatěžováno plochou lisovací čelistí, která se pohybuje ve směru osy z konstantní rychlostí v. V předloženém příspěvku je počítána silová odezva. To nám umožní porovnávat numerické výsledky s reálnými daty z reologických experimentů. Je odvozena variační formulace úlohy a transformována do válcových souřadnic. Dále jsou prezentovány numerické výsledky.

NUMERISCHE BEWERTUNG EINES RHEOLOGISCHEN EXPERIMENTS

Wir betrachten einen viskoelastischen, einfach unterstützten, drehsymmetrischen Körper, der fest mit dem Untergrund verbunden ist. Der Körper wird mit der Fläche eines Presskiefers beschwert, die sich in Richtung der Achse aus der konstanten Geschwindigkeit v bewegt. Im vorliegenden Beitrag wird das Kraftecho berechnet. Dies ermöglicht uns einen Vergleich der numerischen Ergebnisse mit den realen Daten aus den rheologischen Experimenten. Daraus wird eine Variantenformulierung der Aufgabe abgeleitet und in Walzenkoordinaten transformiert. Weiter werden numerische Ergebnisse präsentiert.

NUMERYCZNA OCENA EKSPERYMENTU REOLOGICZNEGO

W artykule rozważane jest wiskoelastyczne, prosto podparty, rotacyjnie symetryczne ciało na stałe połączony z podłożem. Na ciało oddziałuje powierzchnia szczęk prasujących, która porusza się w kierunku osi z stałą prędkością v. W opracowaniu obliczana jest reakcja siłowa. Umożliwia to porównanie wyników numerycznych z realnymi danymi z eksperymentów reologicznych. Określona jest zmienna formuła zadania, która jest transformowana do współrzędnych cylindrycznych. Następnie zaprezentowano wyniki numeryczne.