Optimal Error Estimates for Nonstationary Singularly Perturbed Problems for Low Discretization Orders

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Abstract

We consider an unsteady 1D singularly perturbed convection–diffusion problem. We discretize such a problem by the linear finite element method (FEM) on a Shishkin mesh and by a discontinuous Galerkin method in time. We present optimal a priori error estimates for low order time discretizations.

Keywords: Convection–diffusion; Shishkin mesh; time discontinuous Galerkin method.

Introduction

We focus ourselves on the analysis of the solution of an unsteady linear singularly perturbed convection–diffusion equation. This type of equation can be considered as a simplified model problem to many important problems, especially to Navier–Stokes equations, which describe compressible flow.

We discretize our equation in space by the simple conforming linear finite element method on a Shishkin mesh and then we discretize the resulting system of ordinary differential equations by the discontinuous Galerkin method.

The space discretization of such a problem is a difficult task and it stimulated development of many stabilization methods (e.g. a streamline upwind Petrov–Galerkin (SUPG) method, local projection stabilizations) and layer–adapting techniques (e.g. Shishkin meshes, Bakhvalov meshes). For the complete overview see [6].

Considering the space discretization on Shishkin meshes, we will follow the theory for stationary singularly perturbed problems based on the solution decomposition, which enables us to derive a priori error estimates independent of the diffusion parameter even with respect to the norms (seminorms) of the exact solution, which can be also highly dependent on the diffusion parameter. For the details see [6].

The discontinuous Galerkin (DG) method is a very popular approach for solving ordinary differential equations arising from the space discretization of parabolic problems, which is based on the piecewise polynomial approximation in time. Among important advantages we should mention unconditional stability for arbitrary order, which allows us to solve stiff problems efficiently, and good smoothing property, which enables us to work with inexact or rough data. We should also mention that the DG method is suitable for changes in our computational domain and in computational spaces, which allows us to exploit adaptivity during the computational process. For introduction to the DG time discretization see e.g. [8].

In [1] and [4] the authors study the DG method in time and the local projection stabilization respectively the DG method in space on standard meshes for singularly perturbed problems. The error estimates in these papers contain norms of the exact solutions which go to infinity if the diffusion parameter goes to 0.

There are only few papers dealing with finite elements in space on the special meshes combined with any discretization in time. We should mention [3] and [5], where the backward difference formula (BDF) time discretizations and the θ–scheme are used and a priori error estimates are derived. In [5] the authors also study the DG time discretization and derive suboptimal error estimates.

In contrast to the results in [5], we present a sketch of the proof of optimal error estimates for the DG time discretization for lower convergence orders in $L^\infty(L^2)$ norm.

The main difficulty in proving optimal error estimates for the DG time discretization combined with a space discretization on a Shishkin mesh is the fact that we cannot employ a standard technique of the proof, which is based on the construction of a suitable projection, which enables us to eliminate a discrete time derivative in the error equation, see e.g. [7]. This technique enforces us to do the upper bound of the projection error contained in stationary terms, which depends on a higher time derivative of the exact solution in H^1 seminorm, which depends on the diffusion parameter.

1 Continuous Problem

Let us consider the 1D parabolic singularly perturbed problem

$$
\begin{aligned}
\frac{\partial u}{\partial t}(x,t) - \varepsilon \frac{\partial^2 u}{\partial x^2}(x,t) + b\frac{\partial u}{\partial x}(x,t) + cu(x,t) &= f(x), \quad \forall x \in (0,1), t \in (0,T), \\
u(0,t) = u(1,t) &= 0, \quad \forall t \in (0,T), \\
u(x,0) &= u^0(x), \quad \forall x \in (0,1),
\end{aligned}
\tag{1}
$$

where functions $f \in L^2(0,1)$, $u^0 \in L^2(0,1)$, $0 < \varepsilon << 1$ and functions b and c are sufficiently smooth with $b(x) > \beta > 0$. By substitution in time variable we can achieve

$$
c - \frac{1}{2}\frac{\partial b}{\partial x}(x) \geq c_0 > 0. \tag{2}
$$

Let us define the bilinear form

$$
a(u,v) = \int_0^1 \varepsilon \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \left(b\frac{\partial u}{\partial x} + cu \right) v dx. \tag{3}
$$

To simplify the text we will use the following notation. $(.,.)$ and $\|.\|$ are $L^2(0,1)$ scalar product and norm, $|.|_1$ and $\|.\|_1$ are $H^1(0,1)$ seminorm and norm.

It is possible to show that the solution has in general the boundary layer at $x = 1$. When we assume sufficiently compatible data, we can avoid the existence of interior layers, which enables us to concentrate on the boundary layer only, see [6]. Moreover, it is possible to prove

$$
\left| \frac{\partial^{k+j} u(x,t)}{\partial^k x \partial^j t} \right| \leq C \left(1 + \frac{1}{\varepsilon^k e^{\beta(1-x)/\varepsilon}} \right), \quad \forall k, j \geq 0. \tag{4}
$$

This result shows dependence of space derivatives on ε, which complicates deriving of standard a priori error estimates.

1.1 Discretization

We want to discretize the problem (1) by the standard finite element method on Shishkin meshes in space. This technique allows us to derive a priori error estimates that are independent of ε. We define the parameter

$$\sigma = \frac{5}{2}\frac{\varepsilon}{\beta}\log(N), \tag{5}$$

where N is the number of mesh points. We assume our mesh points are equidistantly distributed in intervals $[0, 1-\sigma]$ and $[1-\sigma, 1]$, with the same number of mesh points in both intervals. Let us define the conforming linear finite element space V_N on our mesh.

To discretize the problem (1) in time we assume the time partition $0 = t_0 < t_1 < \ldots < t_r = T$ with time intervals $I_m = (t_{m-1}, t_m)$, time steps $\tau_m = |I_m| = t_m - t_{m-1}$ and $\tau = \max_{m=1,\ldots,r} \tau_m$. We denote the function values at the nodes as $v^m = v(t_m)$. To be able to use the Galerkin type of discretization we denote the space of piecewise polynomial functions

$$V_N^\tau = \{v \in L^2(0,T,V_N) : v|_{I_m} = \sum_{j=0}^{q} v_{j,m} t^j,\ v_{j,m} \in V_N\}. \tag{6}$$

For the functions from such a space we need to define the values at the nodes of time partition

$$v_\pm^m = v(t_m\pm) = \lim_{t\to t_m\pm} v(t) \tag{7}$$

and the jumps

$$\{v\}_m = v_+^m - v_-^m. \tag{8}$$

Definition 1 *We say that the function $U \in V_N^\tau$ is the approximate solution to the problem (1) if*

$$\begin{aligned} \int_{I_m} (U', v) + a(U, v)dt + (\{U\}_{m-1}, v_+^{m-1}) &= \int_{I_m} (f, v)dt, \\ &\quad \forall v \in V_N^\tau,\ \forall m = 1, \ldots, r \\ (U_-^0, v) &= (u^0, v) \quad \forall v \in V_N. \end{aligned} \tag{9}$$

2 Error Analysis

We define the weighted norm

$$\|v\|_\varepsilon^2 = \varepsilon|v|_1^2 + \|v\|^2, \quad \forall v \in H^1(0,1). \tag{10}$$

It is possible to show that

$$a(v,v) \geq \min(c_0, 1)\|v\|_\varepsilon^2 \geq 0. \tag{11}$$

2.1 Stationary Problem

In this part we want to go through some well known results for the singularly perturbed problems (for the details see [6]). Let us assume a related stationary problem

$$a(u,v) = (f^*, v), \quad \forall v \in H_0^1(0,1), \tag{12}$$

with some $f^*: L^2(0,1)$, and the corresponding discrete finite element problem on the layer–adapted mesh. Let us define the Ritz projection $R: H_0^1(0,1) \to V_N$ satisfying

$$a(u-Ru,v)=0, \quad \forall v \in V_N. \tag{13}$$

It is possible to prove following estimates:

$$\begin{aligned} \|u-Ru\|_\varepsilon &\leq CN^{-1}\log(N), && (14)\\ \|u-Ru\| &\leq C(N^{-1}\log(N))^2, && (15)\end{aligned}$$

with C independent of ε. For the proof see e.g. [6].

2.2 Radau Quadrature

Let us define the Radau quadrature on each interval I_m

$$Q[f]=\sum_{i=0}^{q} w_i f(t_{m,i}), \tag{16}$$

where $t_{m,i}$ are Radau quadrature nodes in I_m with $t_{m,0}=t_m$. Such a quadrature has the algebraic order $2q$ and the coefficients of the quadrature satisfy $0 \leq w_i \leq \tau_m$.

It is possible to express our method (9) by

$$Q[(U',v)]+Q[a(U,v)]+(\{U\}_{m-1},v_+^{m-1})=Q[(f,v)], \quad \forall v \in V_N^\tau. \tag{17}$$

Since the equation for the continuous solution (1) is defined at every point $t \in I_m$, we can see that

$$Q[(u',v)]+Q[a(u,v)]+(\{u\}_{m-1},v_+^{m-1})=Q[(f,v)], \quad \forall v \in V_N^\tau. \tag{18}$$

2.3 Projections

We define the space

$$V^\tau=\{v \in L^2(0,T,H_0^1(0,1)) : v|_{I_m}=\sum_{j=0}^{q} v_{j,m}t^j,\ v_{j,m} \in H_0^1(0,1)\}. \tag{19}$$

We define the time projection $P: C([0,T],H_0^1(0,1)) \to V^\tau$, such that

$$Pu(t)=\sum_{i=0}^{q} \ell_i(t)u(t_{m,i}), \tag{20}$$

where ℓ_i is a Lagrange interpolation basis function for the quadrature node $t_{m,i}$. Since

$$RPu(t)=R\sum_{i=0}^{q} \ell_i(t)u(t_{m,i})=\sum_{i=0}^{q} \ell_i(t)Ru(t_{m,i})=PRu(t), \tag{21}$$

we can see that projections P and R commute. We define the space–time projection $\pi=PR: C(0,T,H_0^1(0,1)) \to V_N^\tau$.

Now, we present some basic approximation properties of our projections P and π.

Lemma 1 *Let u be the exact solution of (1). Then*

$$\sup_{I_m} \|Pu - u\| \leq C\tau^{q+1}, \tag{22}$$

where the constant C does not depend on τ.

The proof can be made by standard arguments. It is an analogy to e.g. [2, Theorem 3.1.5] in Bochner spaces.

Lemma 2 *Let u be the exact solution of (1). Then*

$$\sup_{I_m} \|\pi u - u\| \leq C(\tau^{q+1} + (N^{-1}\log(N))^2), \tag{23}$$

where the constant C does not depend on τ *or* N.

The proof of the lemma follows directly from estimates of the projection P and R and from the triangle inequality.

2.4 Main Result

We are ready to present the main result.

Theorem 1 *Let u be an exact solution of (1) and* $U \in V_N^\tau$ *be its discrete approximation given by (9) with* $q = 0, 1$. *Then*

$$\max_{m=1,\dots,r} \sup_{I_m} \|U - u\| \leq C\left((N^{-1}\log(N))^2 + \tau^{q+1}\right). \tag{24}$$

To prove the theorem we divide the error $U - u$ into projection part $\eta = \pi u - u$ and $\xi = U - \pi u \in V_N^\tau$. Then we subtract the equation for the exact solution (18) from the equation for the discrete solution (17) and we obtain

$$\begin{aligned} &\int_{I_m} (\xi', v) + a(\xi, v)dt + (\{\xi\}_{m-1}, v_+^{m-1}) \\ &= -Q[(\eta', v)] - (\{\eta\}_{m-1}, v_+^{m-1}) - Q[a(\eta, v)]. \end{aligned} \tag{25}$$

Since $Pu(t_{m,i}) = u(t_{m,i})$, we get

$$Q[a(\eta, v)] = \sum_{i=0}^{q} w_i a(Ru(t_{m,i}) - u(t_{m,i}), v) = 0 \tag{26}$$

we need to estimate the rest of the right–hand side only.

Lemma 3 *Let u be an exact solution of (1). Then*

$$Q[(\eta', v)] + (\{\eta\}_{m-1}, v_+^{m-1}) \leq \tau_m C\left(\tau^{q+1} + (N^{-1}\log(N))^2\right) \sup_{I_m} \|v\|, \quad \forall v \in V_N^\tau. \tag{27}$$

The proof of the lemma is rather long and technical and will be published in detail in forthcoming paper.

We can estimate the right–hand side of (25) by Lemma 3. Then we obtain by setting $v = 2\xi$

$$\|\xi_-^m\|^2 - \|\xi_-^{m-1}\|^2 + \|\{\xi\}_{m-1}\|^2 + 2\min(c_0, 1)\int_{I_m} \|\xi\|_\varepsilon^2 dt \tag{28}$$
$$\leq \tau_m C\left(\tau^{q+1} + (N^{-1}\log(N))^2\right) \sup_{I_m} \|\xi\|.$$

We need to deal with the term at the right–hand side. For the case $q = 0$ we know that $\|\xi\|$ is constant with respect to time and we can exchange the last expression by the term $\tau_m C(\tau^{q+1} + (N^{-1}\log(N))^2)\|\xi_-^m\|$. Then it is sufficient to employ Young's inequality and the discrete Gronwall lemma to finish the proof of the theorem. The case $q = 1$ is still quite simple. For $q = 1$ the term $\|\xi\|$ is linear with respect to time and so we can find its supremum at one of the end points of the interval I_m. If the supremum occurs at the point t_m, we can follow the same idea as in the case $q = 0$. If the supremum occurs at the point t_{m-1}, we can divide our term

$$\|\xi_+^{m-1}\| = \|\xi_+^{m-1} - \xi_-^{m-1} + \xi_-^{m-1}\| \leq \|\{\xi\}_{m-1}\| + \|\xi_-^{m-1}\|. \tag{29}$$

Then we need again to employ carefully Young's inequality and the discrete Gronwall lemma to finish the proof of the theorem.

Conclusion

For simplicity, we assume the main result with $q = 0, 1$ only. Nevertheless, the result holds true even for arbitrary $q \geq 0$. Then the proof of the theorem will be more complicated and we will not consider such cases. The result holds also true for consistent stabilization methods and general layer–adapted meshes with a slightly different term describing convergence behavior with respect to space. It is also not important to restrict ourselves to 1D. We can simply extend the results from [3] discussing multidimensional case. The fully general result with complete detailed proofs will be publishes in forthcoming paper.

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OPTIMÁLNÍ ODHADY CHYB NESTACIONÁRNÍCH SINGULÁRNĚ PERTURBOVANÝCH PROBLÉMŮ PRO DISKRETIZACE NÍZKÉHO ŘÁDU

Uvažujeme nestacionární jednodimenzionální singuálrně perturbovaný problém. Diskretizujeme tento problém v prostoru pomocí metody konečných prvků na Shishkinových sítích a v čase pomocí nespojité Galerkinovy metody. Ukážeme optimální apriorní odhady chyb pro časové diskretizace nízkého řádu.

OPTIMALE SCHÄTZUNGEN NICHTSTATIONÄRER SINGULÄR GESTÖRTES PROBLEME FÜR DIE DISKRETIERUNG NIEDERER ORDNUNG

Wir betrachten ein nichtstationäres eindimensionales singulär gestörtes Problem. Wir diskretieren dieses Problem im Raum mit Hilfe der Methode der finiten Elemente auf den Shishkin'schen Netzen und in der Zeit mit Hilfe der nichtkontinuierlichen Galerkin-Methode. Wir zeigen die optiamlen A-priori-Schätzungen von Fehlern für die zeitliche Diskretierung niederer Ordnung.

OPTYMALNE SZACOWANIE BŁĘDÓW NIESTACJONARNYCH OSOBLIWIE ZABURZONYCH PROBLEMÓW DLA DYSKRETYZACJI NISKIEGO RZĘDU

W artykule opisano niestacjonarny jednowymiarowy osobliwie zaburzony problem. Problem ten jest dyskretyzowany w przestrzeni przy pomocy metody elementów skończonych na sieciach Shishkina oraz w czasie przy pomocy nieciągłej metody Galerkina. Pokazano optymalne a priori szacunki błędu dla czasowej dyskretyzacji niskiego rzędu.